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ORIGINAL ARTICLE

An Analytical Solution for the Interface Crack in a Quasi-Brittle Bonding Material

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ABSTRACT

An analytical solution to the problem of an interface crack in the presence of a small-scale process zone in a quasi-brittle bonding material near the crack tip is presented. In the process zone, a quadratic strength criterion of the Mises–Hill type is assumed. The problem of calculating the parameters of the process zone is reduced to a vector functional equation, for which a precise analytical solution can be found using the Wiener–Hopf method. From the solution found, a closed system of transcendental equations is derived for determining the length of the process zone, the phase angle of the load, and the energy release rate in the zone, which form an accessible algorithm for estimating the limit loads that precede propagation of the crack along the interface. Numerical analysis of the developed model is performed, and the possibility of its application in fracture studies of some composite materials is demonstrated.

1 | Introduction

The trends of modern industry towards the use of bimaterials, such as composites, laminates, adhesive or welded joints of dissimilar materials, in the crucial elements of technical products stimulate active research on their reliability and determination of the conditions of their fracture. One of the most probable mechanisms of the bimaterial fracture is the propagation of a crack along the boundary of the bonded materials, therefore, much attention is paid to this phenomenon.

Initial studies of piecewise homogeneous bodies with a crack at a flat interface between two dissimilar materials [1–4] and others within the framework of a fully open crack model with stress-free faces resulted in solutions with spatial oscillations of displacements and prediction of physically impossible multiple mutual penetration of the crack faces at the vicinity of its tips caused by these oscillations. This led to certain difficulties with the correct application of the criteria of linear fracture mechanics [5] to the model of an open interfacial crack and to the development of an alternative model, that is, the model of [6], in which the presence of a region of contact of the faces is prescribed near the tip of the interfacial crack. Studies [6–9] indeed confirmed the elimination of displacement oscillations due to face contact, but under certain loading conditions (e.g., with prevailing tensile loads) predicted extremely small sizes of the contact region, which cast doubt on the assumption of the contact existence. The search for a solution to the above-mentioned problems and the practical needs of the predicted use of bimaterials contributed to the further active development of interfacial crack models and studies on the interfacial fracture.

Over the past decades, a significant amount of experimental and theoretical work has been carried out and important results have been obtained regarding the mechanisms and criteria of interfacial failure. In particular, a well-founded criterion for choosing alternatives between crack propagation along the interface or its kinking into one of the materials of the bimaterial joint has been established, which is based on a comparison of energy release rates for both options of the crack trajectory [10].

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Summary

- A new analytical model of an interfacial crack is proposed.
- In the process zone, a quadratic strength criterion of the Mises–Hill type is assumed.
- The problem is reduced to a vector functional equation.
- Numerical analysis of the developed model is performed.

In the theoretical studies of interfacial fracture, various approaches to the formation of ideas about the mechanisms of crack nucleation and propagation were proposed, among which the concept of the cohesive zone gained considerable popularity. It is based on the experimentally substantiated proposition of the existence of a process zone in front of the crack, in which, due to a high level of stresses and deformations, the interparticle bonds are partially broken with the formation of voids, microcracks, and other defects. In the first, simplest models developed [11–13] in the 60s of the last century for the case of homogeneous materials, the cohesive zone is represented by a mathematical cut on the extension of the crack, the faces of which interact with a constant stress which is tangential to the plastic type fracture or normal for the brittle, and on the faces of the cut have a jump of tangential or normal displacement, respectively. This version of the models, in particular, was used in the works by several authors [14–17] to calculate the parameters of the small-scale process zone in the bonding material at the end of an interfacial crack at a flat interface.

The cohesive zone model is further developed [18] and a method is proposed to take into account the features of the material fracture process in the form of a nonlinear relationship between the forces of traction and the separation of particles in the zone. The possibility of flexibly varying, at the phenomenological level, the law of "traction-separation" laid down in this version of the cohesive zone model contributed to its successful extension to the study of the processes of fracture of various types of materials, in particular composites [19], adhesive joints [20, 21], elastoplastic and viscoelastic materials and polymers [22–24], concrete [25], and so on.

Without going into a detailed analysis of the numerous versions of the Needleman model of the cohesive zone, one can note that due to the complexity of the governing equations, their implementation is mainly carried out by numerical methods. In this regard, there is a need to develop an alternative version of the cohesive zone model, with the necessary simplifications, which would, on the one hand, take into account the nonlinearity of the criterion relations for the stress state in the zone, and on the other hand, provides the possibility of its analytical solution and the means of rapid evaluation of the conditions of interfacial fracture. The current work is based on alternative models of process zones [26, 27], which are based on a combination of models with a quadratic strength criterion of the Mises–Hill type. This paper contains a solution to the problem of an interfacial crack initiation as a result of the formation of a small-scale process zone at the extension of a crack in the bonding material of a piece-homogeneous body. An analytical model of the process zone is developed to predict the propagation of the interfacial crack. The process zone is represented by the segment of discontinuity of the normal and tangential components of the displacement, on which the normal and tangential stresses satisfy the strength criterion of the Mises–Hill type with the limit values of the tangential and normal stresses in the zone as criterion parameters. With the help of Mellin's integral transformation, the problem of calculating the parameters of the process zone is reduced to the Wiener–Hopf vector equation with respect to the transformants of stress components and displacement gradients, for which an exact analytical solution has been found. From the obtained solution, transcendental equations are derived for determining the length of the process zone, the phase angle of the load and the energy release rate in the zone, which form an accessible algorithm for estimating the critical loads that precede the propagation of the crack along the interface.

2 | Problem Statement

We consider the problem of the conditions for the initiation of an interfacial crack located on the interface plane of two different elastic homogeneous isotropic materials with Young's moduli E_1, E_2 and Poisson's ratios ν_1, ν_2 respectively. Considering the fracture toughness of the bonding material (adhesive) to be significantly lower compared with the fracture toughness of the bonded materials (adherents), in accordance with the basic principles of fracture mechanics, we assume that the crack propagation is preceded by the formation of a process zone in the bonding material near the tip of the crack. In this regard, we arrive at the problem of calculating the parameters of the process zone in a thin adhesive layer for the case of plane deformation.

Due to the quasi-brittle nature of the adhesive fracture in the process zone, significant tensile and shear deformations simultaneously occur, therefore, neglecting the thickness of the adhesive layer, in accordance with the cohesive zone model [11–13] as a mathematical cut in which both the normal and the tangent displacements have a jump. Beyond the process zone, the influence of a thin layer of adhesive is neglected and the condition of an ideal bonding of adherends is used in the form of displacement and stresses continuity at the interface. For the normal and tangential components of the stress on the discontinuity line of displacement, we choose the strength criterion of the Mises–Hill type

$$\left(\frac{\sigma_\theta}{\sigma_0}\right)^2 + \left(\frac{\tau_{r\theta}}{\tau_0}\right)^2 = 1, \tag{1}$$

where σ_0, τ_0 are the tensile and shear strengths of the bonding material, which are experimentally determined as the average values of the normal and tangential stresses in the zone under the corresponding loading mode of a homogeneous material. This choice is due to the need to take into account two important factors of interfacial fracture: first, the inevitable mixing of load modes in the process zone, which is a consequence of the impossibility of separating symmetric and skew-symmetric stress fields near the tip within the framework of the model of an open interfacial crack [2–4, 28]; secondly, the quasi-brittle

mechanism of fracture of the bonding material, which assumes the commensurable contribution of tensile and shear stresses in the zone. The justification for using the Mises–Hill type strength criterion in the form (1) is its validity, demonstrated in studies of models of process zones near the tips of cracks in orthotropic plates under biaxial loading [27] and in some variants of the theory of interfacial failure of composites [29–32].

The quasi-brittle nature of adhesive fracture implies the small-scale of the process zone in it, which means the small zone length, l, compared with the characteristic scale of the model L, which is relevant for the stress-strain state near the crack tip. The length of the crack, the distance to the point of application of the load or to the nearest border of the piece-homogeneous body at hand, and so on, can serve as such a scale. Accordingly, limiting ourselves to the study of the stress-strain state in the vicinity of the zone, we come to the problem for a discontinuity line of displacement of finite length, which extends from the tip of a semi-infinite interfacial crack along interface of two elastic half-planes (Figure 1). The small scale of the process zone also allows us to use Prandtl's method of matched asymptotic expansions [33] to formulate the condition at infinity in the form of the requirement of matching the sought solution of the problem for $l \ll r \ll L$ with the asymptotics of the solution of a similar problem of the theory of elasticity without a discontinuity line, which is known from works [2–4, 28].

Taking into account the condition (1) and considering the faces of the crack to be free of loads, we proceed to the static boundary value problem of the theory of elasticity with boundary conditions

$$\theta = -\pi \cup \pi: \sigma_\theta = \tau_{r\theta} = 0; \tag{2}$$

$$\theta = 0: \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0; \tag{3}$$

$$\theta = 0, r < l: \sigma_\theta = \sigma_0 \cos\psi(r), \tau_{r\theta} = \tau_0 \sin\psi(r);$$

$$\theta = 0, r > l: \langle u_\theta \rangle = \langle u_r \rangle = 0; \tag{4}$$

$$\begin{aligned} \theta &= 0, r \to \infty: \sigma_\theta \sim C_1 r^{-1/2+i\omega} \\ &+ \overline{C}_1 r^{-1/2-i\omega}, \tau_{r\theta} \sim C_2 r^{-1/2+i\omega} + \overline{C}_2 r^{-1/2-i\omega}; \end{aligned} \tag{5}$$

where $\langle f \rangle$ denotes a jump in the value of f; $\psi(r)$ is the phase angle of the stress in the process zone, which in the future, due to its small size and to simplify the solution of the problem, is considered to be constant and equal to its average value ψ;

$$C_1 = e' K L^{-i\omega}, C_2 = -iC_1, K = K_1 + iK_2;$$

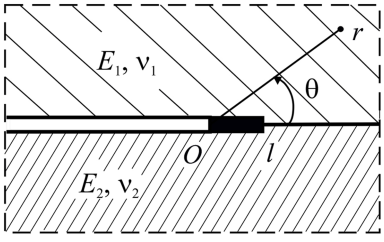


FIGURE 1 | Computational scheme of the problem.

dash above C_1, C_2 and other values means complex conjugation operation; K is the normalized complex stress intensity factor near the crack tip [28];

$$e' = \frac{1}{\sqrt{8\pi \cdot (1-\beta^2)}}, \omega = \frac{1}{2\pi} \ln \frac{1-\beta}{1+\beta},$$
$$\beta = \frac{(1+e\kappa_2)-(e+\kappa_1)}{(1+e\kappa_2)+(e+\kappa_1)};$$

$$e = \frac{E_1}{E_2} \frac{1+\nu_2}{1+\nu_1}, \kappa_{1(2)} = 3 - 4\nu_{1(2)}$$

(β is the so-called second Dundurs parameter, which is one of the measures of the difference between the bonded materials). It is assumed that the condition $\cos\psi \geq 0$ holds, which is necessary for the normal stress in the process zone being tensile.

At the end of the process zone, the asymptotics of the singular part of the solution of the homogeneous boundary value problem for the semi-infinite discontinuity line of displacements at the rectilinear interface of two different elastic materials, similar to an interfacial crack, should hold. In particular, the stresses should satisfy the following conditions [28]:

$$r \to l + 0, \sigma_\theta(r, 0) + i\tau_{r\theta}(r, 0) = \frac{k}{\sqrt{2\pi}} (r-l)^{-1/2+i\omega}, \tag{6}$$

where $k = k_1 + ik_2$ is the local SIF at the end of the zone.

3 | Solving the Problem

The solution of the formulated boundary value problem can be found in the form of the sum of the following two problems. The first differs from the initial problem in that instead of the first pair of conditions in (4) the following conditions are used:

$$\theta = 0, r < l: \sigma_\theta = \sigma_0 \cos\psi - C_1 r^{-1/2+i\omega} - \overline{C}_1 r^{-1/2-i\omega},$$

$$\tau_{r\theta} = \tau_0 \sin\psi - C_2 r^{-1/2+i\omega} - \overline{C}_2 r^{-1/2-i\omega}, \tag{7}$$

and at infinity the stresses decrease as $o(1/r)$. The second problem is a similar problem without a discontinuity line, the solution of which is known [2–4, 28], so the problem lays in solving the first problem.

To construct the solution of the first problem, one can apply the integral Mellin transformation of the form $f^*(p, \theta) = \int_0^\infty f(r, \theta) r^p dr$ to equilibrium equations, deformation compatibility conditions, Hooke's law and boundary conditions (2)–(3). Taking into account the conditions (7) and the second pair of conditions (4), we arrive at the Wiener–Hopf vector functional equation of the first problem in the band $-\varepsilon_1 < \mathrm{Re}(p) < \varepsilon_2$ (ε_1, ε_2 are sufficiently small positive numbers), containing the imaginary axis:

$$\mathbf{\Phi}^+(p) + \mathbf{F}(p) = -A \tan p\pi G_0(p) \mathbf{Q}(p) \mathbf{\Phi}^-(p), \tag{8}$$

$$\mathbf{\Phi}^+(p) = \begin{pmatrix} \Phi_1^+(p) \\ \Phi_2^+(p) \end{pmatrix} = \int_1^\infty \begin{pmatrix} \sigma_\theta(\rho l, 0) \\ \tau_{r\theta}(\rho l, 0) \end{pmatrix} \rho^p d\rho,$$