Journal of Mathematical Sciences, Vol. 201, No. 1, August, 2014

INVESTIGATION OF THE INFLUENCE OF PLASTICITY OF MATERIALS ON THE STRENGTH OF A COMPOSITE JOINT

M. V. Dudyk and Yu. V. Dikhtyarenko

UDC 539.375

Under the conditions of plane deformation, by the Wiener–Hopf method, we perform the numerical analysis of the process zone in the vicinity of the tip of an interface crack that propagating from an angular point of the interface of two different materials one of which is plastic. The zone is modeled by a lateral segment of discontinuity of tangential displacements and by a segment of discontinuity of normal displacements on the interface. We study the influence of the formation of a plastic zone on the conditions of crack start.

Experimental investigations of cracks reveal a complex structure of the material near the end of the crack and, in particular, the presence of a developed plastic zone in the vicinity of the crack tip and a process zone much smaller than the indicated plastic zone [6–8]. There are few analytic models taking into account the simultaneous existence of both zones. In numerous works [3–5, 11], it is proposed to use the so-called "trident" model for the solution of some symmetric problems of fracture mechanics. In this model, it is assumed that the process zone develops in two stage. In the first stage, two narrow plastic strips modeled by the segments of discontinuity of tangential displacement symmetrically propagate from the crack tip at a certain angle to its initial direction. Since their appearance does not remove the stress concentration in the vicinity of the tip, in the next stage, we observe the formation of process zone on the continuation of the crack. This zone is modeled by a segment of discontinuity of normal displacements.

In the present work, we compute the initial process zone under the conditions of plane strain in the vicinity of the tip of an interface crack propagating from a corner point of the broken interface of two materials, one of which is elastoplastic and the other is elastic. The posed problem is reduced to the solution of two problems. In the first problem, we compute a narrow plastic strip propagating in the plastic material from the crack tip at a certain angle to the interface. In the second problem, we solve the problem of the process zone appearing on the interface after the formation of the lateral plastic strip. The obtained results are compared with the results of the solution of the problem of interface process zone without regard for the plasticity of materials of the composite [1]. Note that a fairly large number of investigations (many of them have been carried out within the last decade [12–16] are devoted to the influence of the plasticity of materials on the interface strength in the case of a crack located on the plane interface of materials.

Computation of a Lateral Plastic Strip

Under the conditions of plane strain, we consider the problem computing the initial plastic zone in the vicinity of the tip of an interface crack propagating from a corner point of the interface of two homogeneous isotropic media with Young's moduli E_1 and E_2 and Poisson's ratios ν_1 and ν_2.

Uman State Pedagogical University, Uman, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 56, No. 1, pp. 71–83, January–March, 2013. Original article submitted March 4, 2012.

 © 2014 Springer Science+Business Media New York

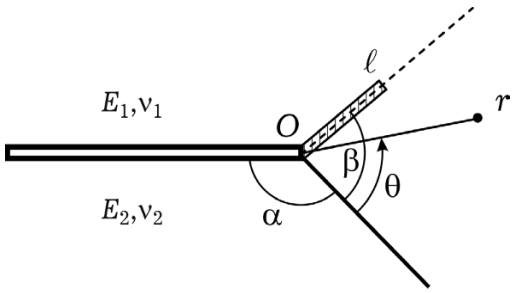


Fig. 1. Computational scheme of the problem of determination of the initial plastic zone.

Assume that only the first material is elastoplastic and, hence, according to the hypothesis of localization [10], a narrow plastic strip propagating from the crack tip at an angle of β to the interface is formed in this material under low loads. According to the Leonov–Panasyuk–Dugdale model, we simulate this plastic zone by a notch subjected to the action of constant tangential stress τ_1. It is assumed that solely the tangential component of displacements may have jumps on the notch. The length of the line of discontinuity ℓ and its orientation are determined in the process of solution of the problem.

In the initial stage of development, the length ℓ of the plastic strip (zone) is much smaller than the length L of the crack and all other sizes of the body and, for this reason, it is assumed that the body can be regarded as a piecewise homogeneous plane formed by two wedges with interface in the form of the sides of a corner, a line of discontinuity of finite length originating at the vertex of this corner and propagating into the first material, and a semiinfinite crack propagating along one of the sides (Fig. 1).

The boundary conditions at infinity are determined by the asymptotic solution in the vicinity of the corner point of a similar problem without the line of discontinuity, which corresponds to the roots of its characteristic equation λ_i whose real parts lie in the interval $(-1,0)$. The characteristic equation has the form [1]

$$D(\lambda) = 0, \tag{1}$$

where

$$D(\lambda) \equiv -(1+\kappa_1)^2 t_1 - 4(1+\kappa_1)(e-1)t_1 t_2 - e^2(1+\kappa_2)^2 t_3 + 4(e-1)^2 t_1 t_3$$

$$+ 4e(1+\kappa_2)(e-1)t_3 t_4 + 2e(1+\kappa_2)(1+\kappa_1)t_5,$$

$$t_1 = (\lambda+1)^2 \sin^2\alpha - \sin^2(\lambda+1)\alpha, \qquad t_2 = \sin^2(\lambda+1)(2\pi-\alpha),$$

$$t_3 = (\lambda+1)^2 \sin^2\alpha - \sin^2(\lambda+1)(2\pi-\alpha), \qquad t_4 = \sin^2(\lambda+1)\alpha,$$

$$t_5 = t_1 + \sin(\lambda+1)\alpha \sin 2\lambda\pi \cos(\lambda+1)(2\pi-\alpha),$$

$$e = \frac{1+\nu_2}{1+\nu_1}\frac{E_1}{E_2}, \qquad \kappa_{1(2)} = 3 - 4\nu_{1(2)}.$$

The interval $(-1,0)$ may contain several roots of Eq. (1) two of which may be complex conjugate [1]. These roots correspond to physically incorrect space oscillations of the stresses and displacements observed in approaching the vertex of the corner. In what follows, we take $\operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 < \lambda_3$.

Under the assumption that the faces of the crack are free of loads and that the plastic strip is modeled by a segment of discontinuity of the tangential displacements, we arrive at the static boundary-value problem of the theory of elasticity with the following boundary conditions:

$$\theta = -\alpha: \quad \sigma_\theta = \tau_{r\theta} = 0,$$

$$\theta = 2\pi - \alpha: \quad \sigma_\theta = \tau_{r\theta} = 0,$$

$$\theta = 0: \quad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0, \quad \langle u_r \rangle = \langle u_\theta \rangle = 0,$$

$$\theta = \beta: \quad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0, \quad \langle u_\theta \rangle = 0,$$

$$\theta = \beta, \quad r < \ell: \quad \tau_{r\theta} = \tau_1,$$

$$\theta = \beta, \quad r > \ell: \quad \langle u_r \rangle = 0,$$

$$\theta = \beta, \quad r \to \infty: \quad \tau_{r\theta} = \sum_i C_i F(\lambda_i, \beta) r^{\lambda_i} + o\left(\frac{1}{r}\right).$$

Here, $\langle f \rangle$ is a jump of f,

$$F(\lambda,\beta) = \frac{X}{(1+\kappa_1)\Delta}\{[e^2(1+\kappa_2)^2 - e(1+\kappa_1)(1+\kappa_2) - 4(e-1)^2 t_1$$

$$- 4e(1+\kappa_2)(e-1)t_3]h_1 + (1+\kappa_1)[2t_1 h_2 + e(1+\kappa_2)h_3]\},$$

$$X = \sqrt{\frac{(e+\kappa_1)}{8\pi(1+e\kappa_2)}},$$

$$\Delta = 2(1+\kappa_1)t_6 - e(1+\kappa_2)t_7 + 4(e-1)(\lambda \sin(\lambda+2)(2\pi-\alpha)\cdot t_8 + \sin\lambda\alpha \cdot t_9),$$

$$h_1 = 2\lambda(\lambda+2)\sin\alpha\sin\beta\cos(\lambda+1)(2\pi-\alpha-\beta)$$

$$+ (\lambda+2)\sin\lambda(2\pi-\alpha)\sin(\lambda+2)\beta - \lambda\sin(\lambda+2)(2\pi-\alpha)\sin\lambda\beta,$$

$$h_2 = -2(e-1)\big[(\lambda+2)\sin(\lambda+1)(2\pi-\alpha+\beta)\sin(\alpha+\beta)$$

$$+2\sin[(\lambda+2)(2\pi-\alpha)-\beta]\sin(\lambda+1)\beta\big]-(1+\kappa_1)\cos(\lambda+2)(2\pi-\alpha-\beta),$$

$$h_3 = 2t_1\cos(\lambda+2)(2\pi-\alpha-\beta)+[(\lambda+2)\sin(\lambda+2)(\alpha+\beta)-\lambda\sin\lambda(\alpha+\beta)]\sin 2\lambda\pi,$$

$$t_6 = \lambda\sin\alpha\cos(\lambda+1)(2\pi-\alpha)-\sin\lambda\alpha\cos 2\lambda\pi,$$

$$t_7 = (\lambda+2)\sin\lambda(2\pi-\alpha)-\lambda\sin(\lambda+2)(2\pi-\alpha),$$

$$t_8 = (\lambda+2)\sin^2\alpha-\sin^2(\lambda+1)\alpha,$$

$$t_9 = (\lambda+2)\sin(\lambda+1)(2\pi-\alpha)\sin(\lambda+1)\alpha-\sin(\lambda+2)(2\pi-\alpha)\sin(\lambda+2)\alpha.$$

The boundary condition imposed at infinity contains arbitrary constants C_i characterizing the intensity of the external force field and assumed to be given in the analyzed problem.

In the solution of the problem, we use the asymptotics

$$\tau_{r\theta}(r,\beta)\sim\frac{k_{\mathrm{II}}}{\sqrt{2\pi(r-\ell)}} \quad \text{as} \quad r\to\ell+0$$

(k_{II} is stress intensity factor at the tip of the plastic strip), which follows from the asymptotic solution of the homogeneous problem of the theory of elasticity for a semiinfinite straight line of discontinuity of the tangential displacements in a homogeneous material.

The solution of the formulated boundary-value problem is obtained by the Wiener–Hopf method in combination with the Mellin integral transformation in exactly the same way as for the problem of a lateral plastic strip at the end of a crack on the plane interface of two different materials [2]. This solution enables us to deduce the transcendental equations for the determination of the relative length of the plastic strip $x=\ell/L$. The direction of propagation of the strip is determined from the condition of maximum of its length. Depending on the existing roots of the characteristic equation (1) in the strip $-1<\mathrm{Re}\,\lambda<0$, we have the following situations:

(a) if the characteristic equation has two or three real roots, the it is associated with the equation for x of the form

$$\sum_{i=1}^{3} n_i F(\lambda_i,\beta)N(\lambda_i,\beta)x^{\lambda_i} = \frac{2}{\tau\sqrt{\pi}\,I(0,\beta)}, \tag{2}$$

where

$$n_i = \frac{C_i}{C_1}\frac{L^{\lambda_i}}{L^{\lambda_1}}, \qquad \tau = \frac{C_1 L^{\lambda_1}}{\tau_1};$$

(b) if the characteristic equation has two complex-conjugate roots ($\lambda_1 = \bar{\lambda}_2$, $C_1 = \bar{C}_2 = (K_I + iK_{II})/\sqrt{2\pi}\, L^{i\operatorname{Im}\lambda_1}$) and one real root (or no real root), then the equation for x takes the form

$$x^{\operatorname{Re}\lambda_1} \cos(\operatorname{Im}\lambda_1 \cdot \ln x + \varphi + \psi + \xi) + n_3 M x^{\lambda_3} = \frac{S}{\tau}, \tag{3}$$

where

$$\varphi = \arg(F(\lambda_1,\beta)), \qquad \xi = \arg N(\lambda_1,\beta), \qquad \psi = \arctan n,$$

$$n = \frac{K_{\mathrm{II}}}{K_{\mathrm{I}}}, \qquad \tau = \frac{|K| L^{\operatorname{Re}\lambda_1}}{\sqrt{2\pi}\,\tau_1}, \qquad n_3 = \frac{\sqrt{2\pi}\, C_3}{|K|} \frac{L^{\lambda_3}}{L^{\operatorname{Re}\lambda_1}},$$

$$M = \frac{F(\lambda_3,\beta)N(\lambda_3,\beta)}{2|F(\lambda_1,\beta)||N(\lambda_1,\beta)|}, \qquad S = \frac{1}{\sqrt{\pi}|F(\lambda_1,\beta)||N(\lambda_1,\beta)|I(0,\beta)},$$

$K = K_{\mathrm{I}} + iK_{\mathrm{II}}$ is the complex stress intensity factor and ψ is the phase angle that characterizes the degree of mixing of the loading modes. In Eqs. (2) and (3), we use the following common notation:

$$N(\lambda,\beta) = \frac{\lambda\Gamma(\lambda)}{(\lambda+0.5)\Gamma(\lambda+0.5)I(\lambda,\beta)}, \qquad I(\lambda,\beta) = \exp\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\ln G(it,\beta)}{1+\lambda+it}\,dt\right],$$

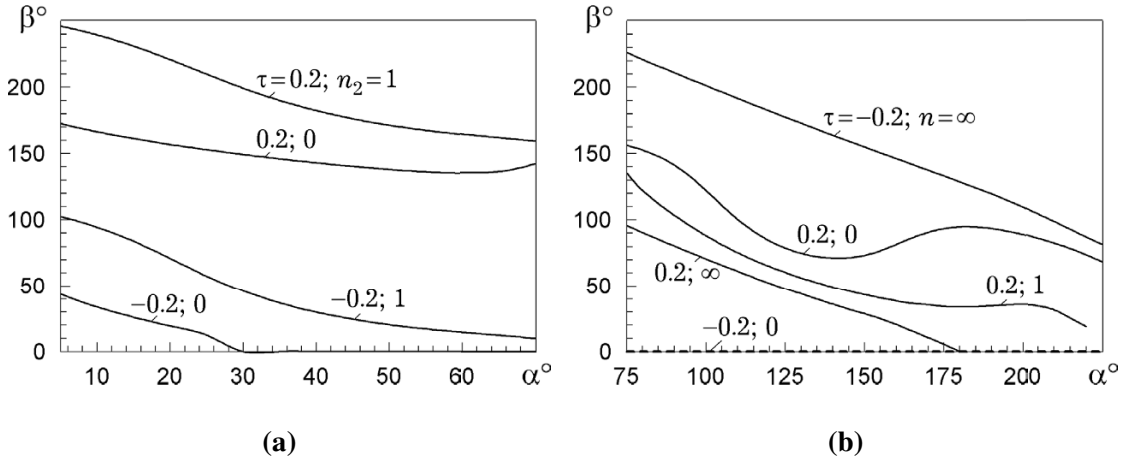
$$G(p,\beta) = \frac{-D_1(p,\beta)}{D(-1-p)} \frac{\cos p\pi}{\sin p\pi},$$

$$\begin{aligned} D_1(p,\beta) = {} & \Delta_1\Delta_2(1+\kappa_1)(e(1+\kappa_2)-(1+\kappa_1)) - (\Delta_2\Delta_3 - \Delta_4\Delta_5)(e^2(1+\kappa_2)^2 \\ & - 4(e-1)^2\Delta_1 - 4e(e-1)(1+\kappa_2)\sin^2 p\alpha) \\ & - 4(e-1)(1+\kappa_1)\Delta_1(\Delta_4 \sin 2p\beta + \Delta_2 \sin^2 p\beta) \\ & + e(1+\kappa_2)(1+\kappa_1)(2\Delta_4\Delta_6 - \Delta_2\Delta_7), \end{aligned}$$

$$\Delta_1(p) = p^2\sin^2\alpha - \sin^2 p\alpha, \qquad \Delta_2(p) = p\sin 2(\alpha+\beta) - \sin 2p(2\pi-\alpha-\beta),$$

$$\Delta_3(p) = p^2\sin^2\beta - \sin^2 p\beta, \qquad \Delta_4(p) = p^2\sin^2(\alpha+\beta) - \sin^2 p(2\pi-\alpha-\beta),$$

$$\Delta_5(p) = p\sin 2\beta + \sin 2p\beta,$$



(a) **(b)**

Fig. 2. Dependences of the angle β of inclination of the plastic strip on the interfacial angle α.

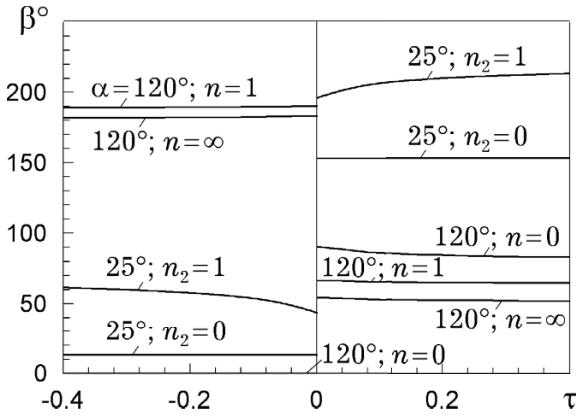


Fig. 3. Dependences of the angle of inclination β of the plastic strip on the dimensionless loading parameter τ.

$$\Delta_6(p) = p\sin\alpha\cos(\alpha+2\beta) + \sin p\alpha\cos p(\alpha+2\beta),$$

$$\Delta_7(p) = p^2\sin\alpha\sin(\alpha+2\beta) - \sin p\alpha\sin p(\alpha+2\beta),$$

where $\Gamma(z)$ is the Euler gamma function and the function $D(z)$ is defined in (1).

In Figs. 2 and 3, we present the plots of the dependences of the angle of inclination of the plastic strip on the interfacial angle and on the load [whose value is characterized by a dimensionless parameter τ and configuration is characterized by the factors n_i and the parameter of mixing of loading modes n in the equations for length (2) and (3)]. In all our numerical calculations, we take $E_1/E_2 = 0.25$, $\nu_1 = \nu_2 = 0.3$, and $n_3 = 0$. Moreover, in what follows, all figures marked by (a) correspond to the case of real roots of Eq. (1), whereas the figures marked by (b) correspond to the case of the complex roots of Eq. (1).

As follows from the calculations of the angle β, the direction of the strip weakly depends on the load but strongly depends on the interfacial angle and the configuration of the load. For certain loading modes, the plastic zone (strip) propagates along the interface of the media $(\beta = 0°)$.

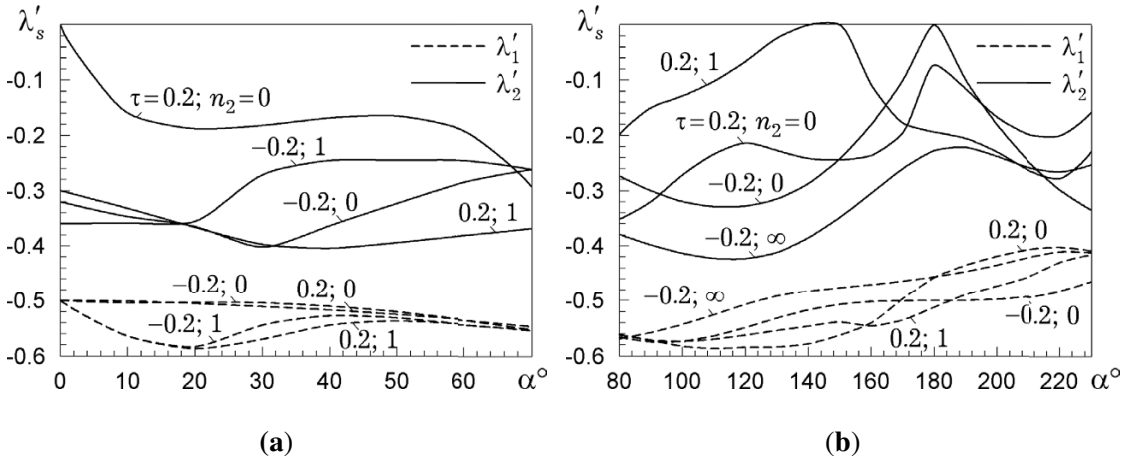


Fig. 4. Dependences of the singularity index of stresses λ'_s on the interfacial angle α.

According to the general principles of the behavior of stresses near concentrators, the behavior of stresses near the tip of an interface crack observed after the formation of the plastic strip can be described as follows: $\sum_s f(\theta,\lambda'_s)r^{\lambda'_s}$, where λ'_s are the roots of the equation $D_1(-1-\lambda'_s,\beta)=0$ from the interval $(-1,0)$.

Since these roots were found to exist, this corresponds to the preservation of stress concentration in the vicinity of the crack tip. The results of evaluation of the roots (Fig. 4) show that the formation of the plastic strip changes the character of the stress-strain state. Thus, in particular, the space oscillations of stresses and displacements are removed and the degree of stress concentration in the vicinity of the crack tip decreases ($\lambda'_1 > \mathrm{Re}\,\lambda_1$). The explicit form of the functions $f(\theta,\lambda'_s)$ is determined from the solution of the Wiener–Hopf problem by using the inverse Mellin transformation.

Calculation of the Secondary Interface Process Zone

The preservation of the stress concentration in the vicinity of the tip of the interface crack at the corner point of the interface of the materials means that, as the load increases, the development of the plastic zone continues by not only in the form of increase in the geometric sizes of the initial zone but also through the formation of new plastic strips or a process zone, which can be found in a similar way. Under the condition of insufficient adhesive strength of the joint of the materials, the propagation of the process zone is possible along their interface. In this connection, we establish the influence of formation of the lateral plastic strip on the strength of the joint of the composite by finding the initial process zone in the vicinity of the tip of the interface crack propagating from the corner point along the interface of the materials. According to the δ_c-model, we model the process zone by a notch whose faces are attracted by the constant stresses σ_0 [9]. Solely the jumps of the normal component of displacements are admitted on the notch and the jump of the tangential component of displacements is neglected. In the initial stage of development, the length d of the process zone is much smaller than the crack length L, the length of the plastic strip and other sizes of the body. Therefore, we assume that the body is a piecewise homogeneous plane formed by two wedges with interface in the form of the sides of a corner and a semiinfinite interface crack, a semiinfinite line of discontinuity of tangential displacements inclined at an angle β to the interface, and an interfacial line of discontinuity of normal displacements of finite length d propagating from the vertex of this corner (Fig. 5).

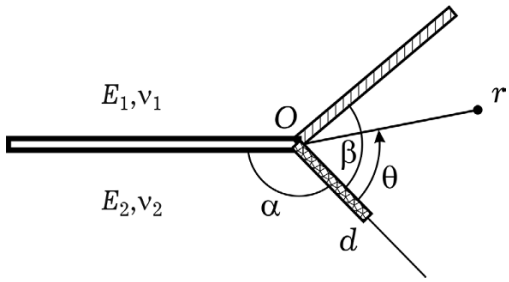


Fig. 5. Computational scheme of the problem of determination of the process zone.

The boundary conditions at infinity are determined by the asymptotic solution in the vicinity of the corner point of a similar problem without the interface process zone obtained in the previous stage. In this connection, with regard for the modeling of the process zone by a segment of discontinuity of normal displacements, we arrive at the static boundary-value problem of the theory of elasticity with the following boundary conditions:

$$\theta = 0: \quad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0, \quad \langle u_r \rangle = 0,$$

$$\theta = -\alpha: \quad \sigma_\theta = \tau_{r\theta} = 0,$$

$$\theta = 2\pi - \alpha: \quad \sigma_\theta = \tau_{r\theta} = 0,$$

$$\theta = \beta: \quad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0, \quad \langle u_\theta \rangle = 0, \quad \tau_{r\theta} = 0,$$

$$\theta = 0, \quad r < d: \quad \sigma_\theta(r,0) = \sigma_0,$$

$$\theta = 0, \quad r > d: \quad \langle u_\theta \rangle = 0,$$

$$\theta = 0, \quad r \to \infty: \quad \sigma_\theta = \sum_s C'_s r^{\lambda'_s} + o\left(\frac{1}{r}\right),$$

where C'_s is determined by the following formulas:

(a) if the characteristic equation (1) has two or three real roots λ_i, then

$$C'_s = \tau\tau_1 \frac{R(-1-\lambda'_s,\beta)}{N(\lambda'_s,\beta)L^{\lambda'_s}} \sum_i n_i F(\lambda_i,\beta) N(\lambda_i,\beta) m(\lambda_i,\lambda'_s) x^{\lambda_i - \lambda'_s};$$

(b) if the characteristic equation has two complex-conjugate roots and one real root (or no real roots), then

$$C_s' = \tau\tau_1 \frac{R(-1-\lambda_s',\beta)}{N(\lambda_s',\beta)L^{\lambda_s'}}[2|F(\lambda_1,\beta)||N(\lambda_1,\beta)||m(\lambda_1,\lambda_s')|x^{\operatorname{Re}\lambda_1-\lambda_s'}$$

$$\times\cos(\operatorname{Im}\lambda_1\cdot\ln x+\varphi+\xi+\psi+\mu_s)+n_3F(\lambda_3)N(\lambda_3)m(\lambda_3,\lambda_s')x^{\lambda_3-\lambda_s'}],$$

where

$$R(p,\beta) = -\frac{q(p,\beta)}{D_1'(p,\beta)}, \qquad m(\lambda_i,\lambda_s') = \frac{\lambda_i}{\lambda_s'(\lambda_i-\lambda_s')}, \qquad \mu_s = \arg m(\lambda_1,\lambda_s'),$$

$$q(p,\beta) = (1+\kappa_1)\{(1+\kappa_1)\Delta_1[\Delta_2\sin(p-1)\beta+2\Delta_8\cos(p-1)\beta]+(2(e-1)\Delta_1$$

$$-e(1+\kappa_2)\Delta_{10})(2p\sin(\alpha+\beta)\Delta_{11}+2\sin p(2\pi-\alpha-\beta)\Delta_{12})-e(1+\kappa_2)(p-1)\Delta_9\Delta_{13}\},$$

$$\Delta_8(p) = p\sin^2(\alpha+\beta)-\sin^2 p(2\pi-\alpha-\beta), \qquad \Delta_9(p) = p\sin 2\alpha-\sin 2p\alpha,$$

$$\Delta_{10}(p) = p\sin^2\alpha-\sin^2 p\alpha, \qquad \Delta_{11}(p) = p\cos p\beta\sin\alpha+\sin p\beta\cos\alpha,$$

$$\Delta_{12}(p) = p\cos p(2\pi-\alpha)\sin\beta-\sin p(2\pi-\alpha)\cos\beta,$$

$$\Delta_{13}(p) = p\sin(\alpha+\beta)\sin p\beta\sin\alpha+\sin p(2\pi-\alpha-\beta)\sin p(2\pi-\alpha)\sin\beta,$$

$$D_1'(p,\beta) = (\Delta_1'\Delta_2+\Delta_1\Delta_2')(e(1+\kappa_2)(1+\kappa_1)-(1+\kappa_1)^2)-(\Delta_2'\Delta_3+\Delta_2\Delta_3'$$

$$-\Delta_4'\Delta_5-\Delta_4\Delta_5')(e^2(1+\kappa_2)^2-4(e-1)^2\Delta_1$$

$$-4e(e-1)(1+\kappa_2)\sin^2 p\alpha)+(\Delta_2\Delta_3-\Delta_4\Delta_5)(4(e-1)^2\Delta_1'$$

$$+4e(e-1)(1+\kappa_2)\alpha\sin 2p\alpha)-4(e-1)(1+\kappa_1)$$

$$\times[\Delta_1'(\Delta_4\sin 2p\beta+\Delta_2\sin^2 p\beta)+\Delta_1(\Delta_4'\sin 2p\beta$$

$$+\Delta_4 2\beta\cos 2p\beta+\Delta_2'\sin^2 p\beta+\Delta_2\beta\sin 2p\beta)]$$

$$+e(1+\kappa_2)(1+\kappa_1)(2\Delta_4'\Delta_6+2\Delta_4\Delta_6'-\Delta_2'\Delta_7-\Delta_2\Delta_7'),$$

$$\Delta_1'(p) = 2p\sin^2\alpha - \alpha\sin 2p\alpha,$$

$$\Delta_2'(p) = \sin 2(\alpha+\beta) - 2(2\pi-\alpha-\beta)\cos 2p(2\pi-\alpha-\beta),$$

$$\Delta_3'(p) = 2p\sin^2\beta - \beta\sin 2p\beta,$$

$$\Delta_4'(p) = 2p\sin^2(\alpha+\beta) - (2\pi-\alpha-\beta)\sin 2p(2\pi-\alpha-\beta),$$

$$\Delta_5'(p) = \sin 2\beta + 2\beta\cos 2p\beta,$$

$$\Delta_6'(p) = \sin\alpha\cos(\alpha+2\beta) + \alpha\cos 2p(\alpha+\beta) - 2\beta\sin p\alpha\sin p(\alpha+2\beta),$$

$$\Delta_7'(p) = 2p\sin\alpha\sin(\alpha+2\beta) - \alpha\sin 2p(\alpha+\beta) - 2\beta\sin p\alpha\cos p(\alpha+2\beta).$$

Assume that a tensile normal stress $\sigma_\theta(r,0) > 0$ acts on the interface of materials and leads to the appearance of tearing strains required for the formation of the process zone.

Near the end of the process zone, the stresses and displacements obey asymptotics in the form of the asymptotically largest solutions of the homogeneous problem of the theory of elasticity for a piecewise homogeneous plane containing a semiinfinite straight line of discontinuity of normal displacements on the plane interface of the media. In particular, the following asymptotic relation is true:

$$\theta = 0, \qquad r \to d+0: \qquad \sigma_\theta \sim \frac{\kappa_1 + 1 + e(1+\kappa_2)}{2(e+\kappa_1)} \frac{k_I}{\sqrt{2\pi(r-d)}},$$

where k_I is the stress intensity factor at the end of the line of discontinuity determined in the process of solution of the problem.

The solution of the formulated boundary-value problem by the Wiener–Hopf method is analogous to the solution of a similar problem for a plastic strip [1]. BY using this solution, we obtain the following transcendental equations for the determination of the relative length of the interface zone $y = d/L$ depending on the existing roots of the characteristic equation (1) in the strip $-1 < \mathrm{Re}\,\lambda < 0$:

(a) if the characteristic equation has two or three real roots, then it is associated with the following equation for y:

$$\sum_s R(-1-\lambda_s',\beta) y^{\lambda_s'} \frac{I(\lambda_s',\beta)}{I_1(\lambda_s',\beta)} \sum_i n_i F(\lambda_i,\beta) N(\lambda_i,\beta) m(\lambda_i,\lambda_s') x^{\lambda_i-\lambda_s'} = \frac{\sigma_0}{\tau\tau_1} S_1(\beta); \tag{4}$$

(b) if the characteristic equation has two complex-conjugate roots and one real root (or no real roots), then the equation for y takes the form

$$\sum_s R(-1-\lambda_s',\beta)y^{\lambda_s'}\frac{I(\lambda_s',\beta)}{I_1(\lambda_s',\beta)}[2|F(\lambda_1,\beta)||N(\lambda_1,\beta)||m(\lambda_1,\lambda_s')|x^{\mathrm{Re}\,\lambda_1-\lambda_s'}$$

$$\times\cos(\mathrm{Im}\,\lambda_1\ln x+\varphi+\xi+\psi+\mu_s)$$

$$+n_3F(\lambda_3)N(\lambda_3)m(\lambda_3,\lambda_s')x^{\lambda_3-\lambda_s'}]=\frac{\sigma_0S_1(\beta)}{\tau\tau_1}, \tag{5}$$

where

$$I_1(\lambda',\beta)=\exp\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\ln G_1(it,\beta)}{1+\lambda'+it}dt\right],\qquad G_1(p,\beta)=\frac{(1+\kappa_1)D_2(p,\beta)}{AD_1(p,\beta)}\frac{\cos p\pi}{\sin p\pi},$$

$$D_2(p,\beta)=(1+\kappa_1)(4(\sin^2\beta-\cos^2 p\beta)\Delta_4+\Delta_2\Delta_{14})\Delta_1+e(1+\kappa_2)(\Delta_2\Delta_3-\Delta_4\Delta_5)\Delta_9,$$

$$\Delta_{14}(p)=p\sin 2\beta-\sin 2p\beta,$$

$$A=\frac{(1+\kappa_1)[e+\kappa_1+1+e\kappa_2]}{2(e+\kappa_1)(1+e\kappa_2)},\qquad S_1(\beta)=\frac{2}{\sqrt{\pi}\,I_1(0,\beta)}.$$

We use the obtained solution to find the opening displacement of the process zone in the vicinity of the crack tip in terms of the jump of normal displacements $\delta=\langle u_\theta(0,0)\rangle$ by analogy with [1]. This gives the following relations:

(a) if Eq. (1) has only real roots in the interval $(-1,0)$, then

$$\delta=-\frac{4(1-\nu_1^2)L}{E_1A\sqrt{\pi G_1(0,\beta)}}\tau\tau_1\sum_s R(-1-\lambda_s',\beta)\frac{I(\lambda_s',\beta)}{I_1(\lambda_s',\beta)}\frac{\lambda_s'}{1+\lambda_s'}y^{\lambda_s'+1}\sum_i n_iF(\lambda_i,\beta)N(\lambda_i,\beta)m(\lambda_i,\lambda_s')x^{\lambda_i-\lambda_s'}; \tag{6}$$

(b) in the presence of complex roots of Eq. (1), we get

$$\delta=-\frac{4(1-\nu_1^2)L}{E_1A\sqrt{\pi G_1(0,\beta)}}\tau\tau_1\sum_s R(-1-\lambda_s',\beta)\frac{I(\lambda_s',\beta)}{I_1(\lambda_s',\beta)}\frac{\lambda_s'}{1+\lambda_s'}y^{\lambda_s'+1}$$

$$\times[2|F(\lambda_1,\beta)||N(\lambda_1,\beta)||m(\lambda_1,\lambda_s')|x^{\mathrm{Re}\,\lambda_1-\lambda_s'}$$

$$\times\cos(\mathrm{Im}\,\lambda_1\ln x+\varphi+\xi+\psi+\mu_s)$$

$$+n_3F(\lambda_3)N(\lambda_3)m(\lambda_3,\lambda_s')x^{\lambda_3-\lambda_s'}], \tag{7}$$

where

$$G_1(0,\beta) = \frac{(1+\kappa_1)}{A\pi}\frac{D_2(0,\beta)}{D_1(0,\beta)},$$

$$D_2(0,\beta) = (1+\kappa_1)g_1(g_2g_8 - 4\cos^2\beta g_4) + e(1+\kappa_2)(g_2g_3 - g_4g_5)g_9,$$

$$D_1(0,\beta) = g_1g_2(1+\kappa_1)[e(1+\kappa_2) - (1+\kappa_1)] - e^2(1+\kappa_2)^2(g_2g_3 - g_4g_5)$$

$$+ e(1+\kappa_2)(1+\kappa_1)(2g_4g_6 - g_2g_7),$$

$$g_1 = \sin^2\alpha - \alpha^2, \quad g_2 = \sin 2(\alpha+\beta) - 2(2\pi - \alpha - \beta), \quad g_3 = \sin^2\beta - \beta^2,$$

$$g_4 = \sin^2(\alpha+\beta) - (2\pi - \alpha - \beta)^2, \quad g_5 = \sin 2\beta + 2\beta,$$

$$g_6 = \sin\alpha\cos(\alpha+2\beta) + \alpha, \quad g_7 = \sin\alpha\sin(\alpha+2\beta) - \alpha(\alpha+2\beta),$$

$$g_8 = \sin 2\beta - 2\beta, \quad g_9 = \sin 2\alpha - 2\alpha.$$

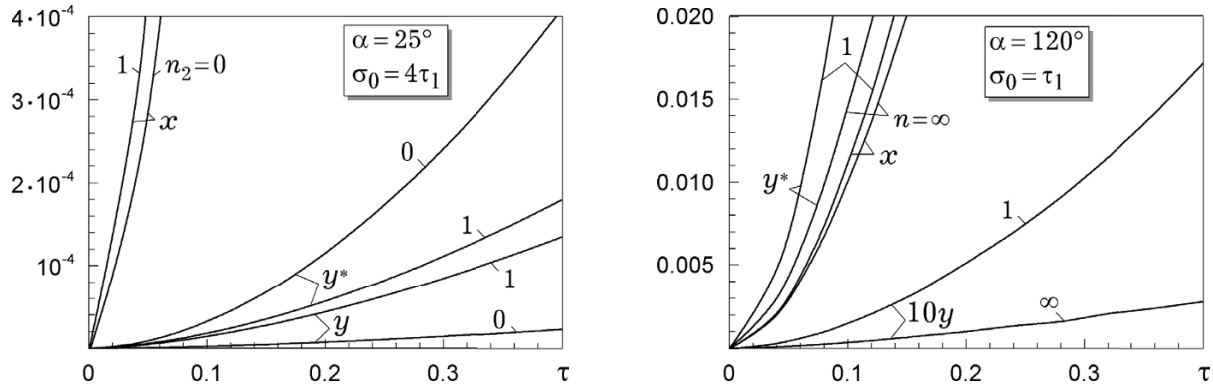
Numerical Results and Their Analysis

The requirements of tensile character of strains on the interface in the vicinity of the vertex of its corner and smallness of the length of interfacial process zone as compared with the length of the lateral plastic strip restrict the range of admissible interfacial angles and configurations of the load for which the investigated model is acceptable. In particular, the results of calculations with parameters for which the plastic strip propagates along the interface are excluded.

The dimensionless parameters τ of the present work and the parameters σ from [1] satisfy the relation $\sigma = \tau_1\tau/\sigma_0$, which enables us to compare the sizes of the interfacial process zone in the presence and in the absence of the lateral plastic strip.

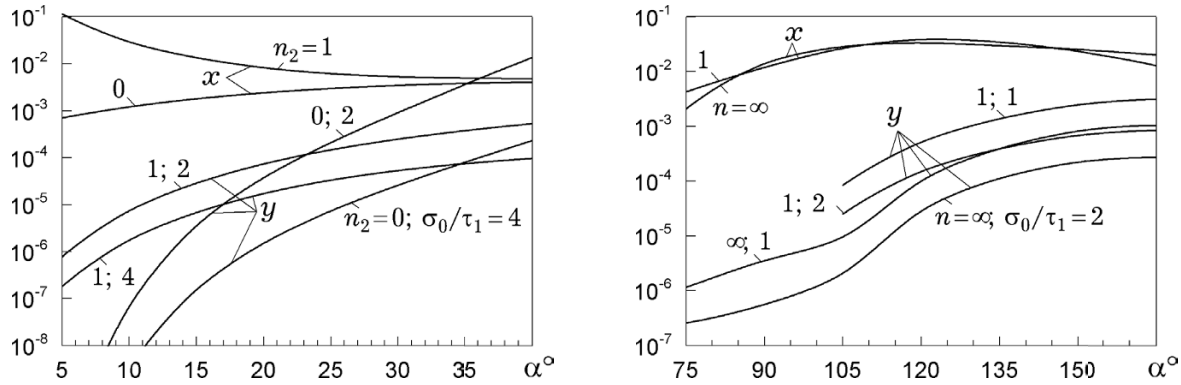
In Fig. 6, we show the results of numerical analyses of the dependences of relative lengths of the plastic strip and process zone on the load for some parameters of the problem and the relative lengths of the interfacial process zone in the absence of lateral plastic strip computed according to the results from [1] with the same parameters. We see that the sizes of the plastic strip and process zone simultaneously increase with the load. In this case, in the presence of plastic strip, the length of the interface zone is much smaller than in the absence of the plastic strip.

Figure 7 shows the influence of the interfacial angle α on the sizes of the process zone. For small angles, the length of the interface zone is much smaller than the length of the lateral plastic strip. However, within certain ranges of interfacial angles, the sizes of these zones are of the same order of magnitude, i.e., the corresponding configuration of the composite joint facilitates the development of the interface process zone and, hence, the process of crack propagation along the interface.



(**a**) (**b**)

Fig. 6. Dependences of the relative lengths of the plastic strip (x), secondary process zone (y), and interface process zone in the absence of the plastic strip (y^*) [1] on the dimensionless parameter of loading τ.



(**a**) (**b**)

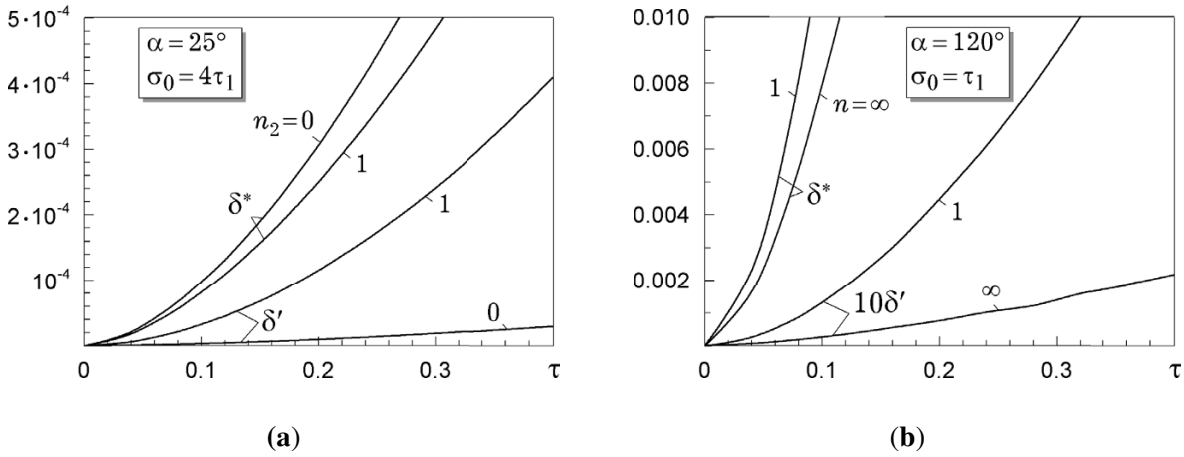
Fig. 7. Dependences of the relative lengths of the plastic strip (x) and secondary process zone (y) on the interfacial angle α for $\tau = 0.2$.

We also note that the length of the interfacial zone increases with α and may exceed the length of the lateral plastic strip. The higher the adhesive strength of the joint characterized by the parameter σ_0, the smaller the sizes of the interfacial zone.

The formation of the lateral plastic strip substantially decreases the crack tip opening displacement, which is illustrated by the plots of dependences of the reduced crack-tip opening displacement

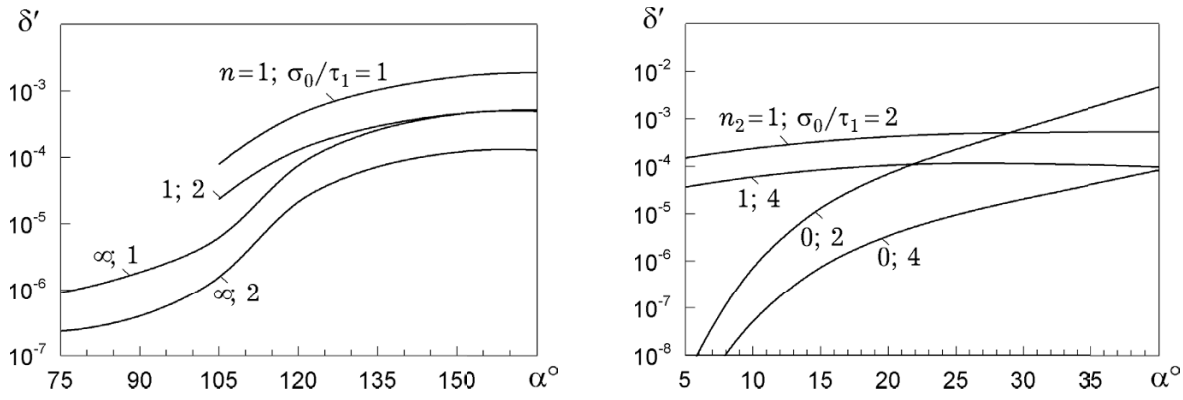
$$\delta' = \frac{E_1}{4(1-\nu_1^2)\sigma_0}\frac{\delta}{L}$$

on the load τ and the interfacial angle in the presence (δ') and in the absence (δ^*) of the strip computed according to relations (6) and (7) and relations taken from [1] and presented in Figs. 8 and 9, respectively. The similarity of these dependences to the analogous dependences of the length of interfacial process zone enables us to reproduce the conclusions made for the process zone in the case of crack opening displacement.



(a) **(b)**

Fig. 8. Dependences of the crack-tip opening displacement in the presence (δ') and in the absence of plastic strip δ^* on the dimensionless loading parameter τ.



(a) **(b)**

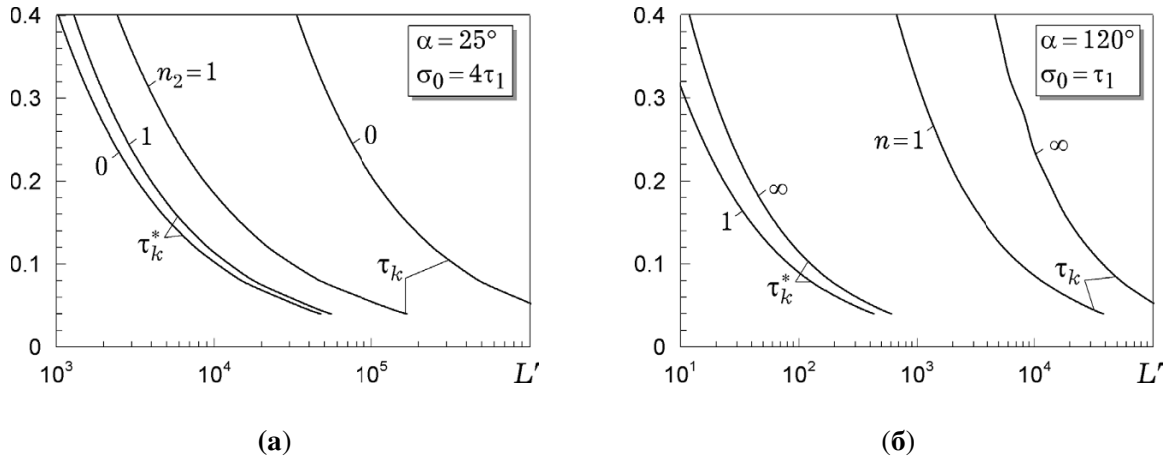
Fig. 9. Dependences of the crack-tip opening displacement on the interfacial angle α for $\tau = 0.2$.

Extrapolating the obtained dependences of the crack opening displacement to fairly high loads, we can find the ultimate load corresponding to the attainment of the critical value δ_n by the crack opening displacement (criterion of critical crack opening displacement (CCOD criterion [9]).

In Fig. 10, we show the results of the numerical analyses of dependences of the dimensionless ultimate loading parameter of τ_k on the reduced crack length

$$L' = \frac{4(1-\nu_1^2)\sigma_0 L}{E_1 \delta_n}$$

in the presence (τ_k) and in the absence $(\tau_k^* = \sigma_0\sigma_k/\tau_1$, [1]) of the lateral plastic strip. The analysis of the plots shows that the formation of the plastic strip substantially increases the ultimate load. Moreover, as could be expected from the physical reasoning, the numerical results quantitatively substantiate the increase in the ultimate load for joints with higher adhesive strength.



(а) **(б)**

Fig. 10. Dependences of the ultimate load in the presence (τ_k) and in the absence (τ_k^*) of the lateral plastic strip on the reduced length of the crack L'.

CONCLUSIONS

By using the Wiener–Hopf method, we determine the solution of the problem of determination of the process zone in the vicinity of the tip of an interface crack propagating from the corner point of the interface of two homogeneous isotropic materials one of which is a plastic material. The zone is represented by a line of discontinuity of tangential displacements in the plastic material inclined to the interface and a much smaller line of discontinuity of normal displacements. We deduce an equation for the determination of their lengths and the crack opening displacement at the corner point and study their dependence on the parameters of the joined materials, value and configuration of load, an the interfacial angle. It is shown that the qualitative dependences of the investigated quantities on the parameters of the composite body the and load in the presence of a plastic strip are similar to the dependences established in [1] in the absence of plastic strip but their numerical values are much lower. This is explained by the fact that the formation of plastic strip unloads the material in the region of the joint by decreasing the level of local stresses. We also compute the ultimate load according to the deformation criterion, investigate its dependence on the crack length and adhesive strength of the joint, and show its increase as a result of the formation of plastic strip.

REFERENCES

1. M. V. Dudyk and Yu. V. Dikhtyarenko, "Development of a process zone from an interface crack at a corner point of an interface of two elastic media," *Mat. Met, Fiz.-Mekh. Polya,* **54,** No. 2, 103–114 (2011); ***English translation:*** *J. Math. Sci.,* **184**, No. 2, 121–135 (2012).
2. A. A. Kaminsky, M. V. Dudik, and L. A. Kipnis, "On the direction of development of a thin fracture process zone at the tip of an interfacial crack between dissimilar media," *Prikl. Mekh.,* **42**, No. 2, 14–23 (2006); ***English translation:*** *Int. Appl. Mech.,* **42**, No. 2, 136–144 (2006).
3. A. A. Kaminskii, L. A. Kipnis, and M. V. Dudik, "Initial development of the process zone near the tip of a crack reaching the interface between dissimilar media," *Prikl. Mekh.,* **40**, No. 2, 74–81 (2004); ***English translation:*** *Int. Appl. Mech.,* **40**, No. 2, 176–182 (2004).
4. A. A. Kaminskii, L. A. Kipnis, and V. A. Kolmakova, "Calculation of the plastic zone at a crack tip with the use of the «trident» model," *Prikl. Mekh.,* **33**, No. 5, 70–76 (1997); ***English translation:*** *Int. Appl. Mech.,* **33**, No. 5, 407–413 (1997).
5. A. A. Kaminskii, L. A. Kipnis, V. A. Kolmakova, and G. A. Khazin, "The use of the «trident» model in the analysis of plastic zones near crack tips and corner points," *Prikl. Mekh.,* **36**, No. 3, 95–100 (2000); ***English translation:*** *Int. Appl. Mech.,* **36**, No. 3, 372–376 (2000).

6. A. A. Kaminskii and S. B. Nizhnik, "Study of the laws governing the changes in the plastic zone at the crack tip and characteristics of the fracture toughness of metallic materials in relation to their structure (survey)," *Prikl. Mekh.,* **31**, No. 10, 3–27 (1995); ***English translation:*** *Int. Appl. Mech.,* **31**, No. 10, 777–798 (1995).
7. A. A. Kaminskii, G. I. Usikova, and E. A. Dmitrieva, "Experimental study of the distribution of plastic strains near a crack tip during static loading," *Prikl. Mekh.,* **30**, No. 11, 69–75 (1994); ***English translation:*** *Int. Appl. Mech.,* **30**, No. 1, 892–897 (1994).
8. I. S. Kohut and H. I. Kalyta, "Evaluation of the sizes of the process zone for quasibrittle notched specimens," *Fiz.-Khim. Mekh. Mater.,* **44**, No. 1, 84–87 (2008); ***English translation:*** *Mater. Sci.,* **44**, No. 1, 97–103 (2008).
9. V. V. Panasyuk, *Limiting Equilibrium of Brittle Bodies with Cracks* [in Russian], Naukova Dumka, Kiev (1968).
10. V. V. Panasyuk and M. P. Savruk, "Model for plasticity bands in elastoplastic failure mechanics," *Fiz.-Khim. Mekh. Mater.,* **28**, No. 1, 49–68 (1992); ***English translation:*** *Mater. Sci.,* **28**, No. 1, 41–57 (1992).
11. G. P. Cherepanov, *General Theory of Failure* [in Russian], *Fiz.-Khim. Mekh. Mater.,* **22**, No. 1, 36–44 (1986); ***English translation:*** *Mater. Sci.,* **22**, No. 1, 32–39 (1986).
12. H. Chai, "Interfacial mixed-mode fracture of adhesive bonds undergoing large deformation," *Int. J. Solids Struct.,* **40,** 6023–6042 (2003).
13. A. G. Evans, J. W. Huthchinson, and Y. Wei, "Interface adhesion: effects of plasticity and segregation," *Acta Mater.,* **47**, 4093–4113 (1999).
14. A. Nakatani, W. J. Drugan, E. van der Giessen, and A. Needleman, "Crack tip fields at a ductile single crystal-rigid material interface," *Int. J. Fracture,* **122**, No. 3–4, 131–159 (2003).
15. V. Tvergaard, "Influence of plasticity on interface toughness in a layered solid with residual stresses," *Int. J. Solids Struct.,* **40,** No. 21, 5769–5779 (2003).
16. V. Tvergaard, "Prediction of mixed-mode interface crack growth using a cohesive zone model for ductile fracture," *J. Mech. Phys. Solids,* **52,** No. 4, 925–940 (2004).