Available online at www.sciencedirect.com

ScienceDirect

Procedia Structural Integrity 68 (2025) 53–58

Structural Integrity Procedia

www.elsevier.com/locate/procedia

European Conference on Fracture 2024

On analytical model of interface crack in bonding quasi-brittle material between distinct elastic media

M.Dudyk[a,*], A. Kaminsky[b], Yu. Reshitnyk[a], Yu. Chornoivan[b]

[a]*Uman State Pedagogical University, Faculty of Phys., Mathem. and Inform., Sadova 2, 20300 Uman, Ukraine*
[b]*S.P. Timoshenko Institute of Mechanics, National Academy of Sciences, Nesterova 3, 03057 Kyiv, Ukraine*

Abstract

Developed is an analytical model of an interfacial crack in the quasi-brittle bonding material with a process zone near its tip. In the process zone, a quadratic strength criterion of the Mises-Hill type is assumed. Using the Mellin integral transformation, the problem of calculating the parameters of the process zone is reduced to a vector functional equation, for which an exact analytical solution was found using the Wiener-Hopf method. From the solution found, a closed system of transcendental equations and relationships is derived for determining the length of the process zone, the phase angle of the load and the energy release rate in the zone, which form an accessible algorithm for estimating the limit loads that precede propagation of the crack along the interface. Numerical analysis of the developed model is performed.

© 2025 The Authors. Published by ELSEVIER B.V.
This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0)
Peer-review under responsibility of ECF24 organizers

Keywords: interface crack; process zone; Mises-Hill criterion; Wiener-Hopf vector equation; crack initiation

1. Introduction

Among the ideas about the mechanisms of nucleation and movement of cracks in piecewise homogeneous bodies, the concept of a cohesive zone has gained considerable popularity. It is based on the experimentally substantiated proposition of the process zone existence in front of the crack tip. In this zone, due to a high level of stresses and deformations, a partial rupture of interparticle bonds occurs with the formation of dislocations, voids, microcracks, and other defects in the structure of the substance. In the first, simplest models developed in the 60s of

* Corresponding author. Tel.: +38-050-353-2998.
E-mail address: dudik_m@hotmail.com

2452-3216 © 2025 The Authors. Published by ELSEVIER B.V.
This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0)
Peer-review under responsibility of ECF24 organizers
10.1016/j.prostr.2025.06.022

the last century by Barenblatt (1962), Dugdale (1960), Leonov and Panasyuk (1959) for the case of homogeneous materials, the cohesive zone was represented by a mathematical cut on the extension of the crack, the edges of which interact with constant stress - tangential for the plastic type of failure or normal for brittle, and on the faces of the cut undergo a jump tangential or normal movement, respectively. This version of the models, in particular, was used to calculate the parameters of the small-scale process zone in the bonding material at the tip of the interfacial crack at the flat interface in works by Kaminskii et al. (1995, 1999) and some other similar works.

The cohesive zone model received further development in the works of Needleman (1987), where a method was proposed to take into account the features of the material destruction process in the form of a non-linear relationship between the forces of cohesion and the separation of particles in the zone. The possibility of flexible variation, at the phenomenological level, of the "traction-separation" law included in this version of the cohesive zone model contributed to its successful extension to the study of the destruction processes of various types of materials, in particular composites, adhesive joints, elastoplastic and viscoelastic materials and polymers, concrete, etc.

However, the implementation of most currently available models in practice faces the need for painstaking numerical calculations in finite element analysis programs. In connection with this, it is desirable to construct such a variant of the process zone model that would provide the possibility of its analytical solution, giving a convenient tool for a quick assessment of the conditions of crack growth.

2. Model description

In this work, under the plane strain condition, an analytical calculation of the process zone parameters is given for the case of a planar interface crack in a quasi-brittle bonding material between two different homogeneous isotropic materials. The process zone is modeled by the displacement discontinuity line where the normal and tangential stresses satisfy the quadratic strength criterion of the Mises-Hill type

$$\left(\sigma_\theta / \sigma_0\right)^2 + \left(\tau_{r\theta} / \tau_0\right)^2 = 1, \tag{1}$$

where σ_0, τ_0 are the tensile and shear strengths of the bonding material.

It is assumed that the length of the zone l is much smaller than the length of the crack L and all other dimensions of the body, and since the stressed state is studied in the vicinity of the zone, the initial problem is reduced to the problem of a discontinuity line of finite length extending from the tip of a semi-infinite interfacial crack along the interface two elastic half-planes (Fig. 1). At infinity, the stitching condition for the desired solution and the known solution of the analogous problem of the theory of elasticity without a discontinuity in the distances $l << r << L$ is formulated (Erdogan, 1963; England, 1965; Rice and Sih, 1965).

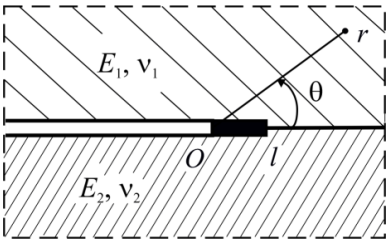


Fig.1. Problem geometry and notation.

Taking into account condition (1) and considering the faces of the crack to be stress-free, one can arrive at the static boundary value problem of the theory of elasticity with the following boundary conditions:

$$\theta = -\pi \bigcup \pi: \quad \sigma_\theta = \tau_{r\theta} = 0; \tag{2}$$

$$\theta = 0: \qquad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0; \tag{3}$$

$$\theta = 0,\ r < l: \quad \sigma_\theta = \sigma_0 \cos\psi(r),\ \tau_{r\theta} = \tau_0 \sin\psi(r); \quad \theta = 0,\ r > l: \quad \langle u_\theta \rangle = \langle u_r \rangle = 0; \tag{4}$$

$$\theta = 0,\ r \to \infty: \quad \sigma_\theta \sim C_1 r^{-0,5+i\omega} + \bar{C}_1 r^{-0,5-i\omega},\ \tau_{r\theta} \sim C_2 r^{-0,5+i\omega} + \bar{C}_2 r^{-0,5-i\omega}; \tag{5}$$

$$C_1 = e' K L^{-i\omega},\ C_2 = -iC_1,\ K = K_I + iK_{II};\ e' = \frac{1}{\sqrt{8\pi \cdot (1-\beta^2)}},\ \omega = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta},\ \beta = \frac{(1+e\kappa_2)-(e+\kappa_1)}{(1+e\kappa_2)+(e+\kappa_1)};$$

$$e = \frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1},\ \kappa_{1(2)} = 3 - 4\nu_{1(2)};$$

$\langle f \rangle$ is the jump of f; $\psi(r)$ is the phase angle of stress in the process zone, which will be considered constant and equal to its average value in what follows ψ; the dash above C_1 and C_2 is the complex conjugate; K is the complex SIF at the tip of the interface crack.

Using the Mellin integral transformation, the formulated boundary value problem (2-5) is reduced to the Wiener-Hopf matrix equation (6) in the strip $-\varepsilon_1 < \mathrm{Re}\, p < \varepsilon_2$ (ε_1, ε_2 are the small positive numbers) which contains the imaginary axis:

$$\mathbf{\Phi}^+(p) + \mathbf{F}(p) = -A \tan p\pi G(p)\mathbf{Q}(p)\mathbf{\Phi}^-(p), \tag{6}$$

$$\begin{pmatrix} \Phi_1^+(p) \\ \Phi_2^+(p) \end{pmatrix} = \int_1^\infty \begin{pmatrix} \sigma_\theta(\rho l, 0) \\ \tau_{r\theta}(\rho l, 0) \end{pmatrix} \rho^p d\rho,\ \begin{pmatrix} \Phi_1^-(p) \\ \Phi_2^-(p) \end{pmatrix} = \frac{E_1}{4(1-\nu_1^2)} \int_0^1 \frac{\partial}{\partial r} \begin{pmatrix} u_\theta \\ u_r \end{pmatrix}_{r=\rho l} \rho^p d\rho,$$

$$\mathbf{F}(p) = \begin{pmatrix} \dfrac{\sigma_0 \cos\psi}{p+1} - \dfrac{C_1 r^{-0,5+i\omega}}{p+0,5+i\omega} - \dfrac{\bar{C}_1 r^{-0,5-i\omega}}{p+0,5-i\omega} \\ \dfrac{\tau_0 \sin\psi}{p+1} - \dfrac{C_2 r^{-0,5+i\omega}}{p+0,5+i\omega} - \dfrac{\bar{C}_2 r^{-0,5-i\omega}}{p+0,5-i\omega} \end{pmatrix},\ A = \frac{(1+\kappa_1)(e+\kappa_1+1+e\kappa_2)}{2(e+\kappa_1)(1+e\kappa_2)},$$

$$G(p) = \frac{4(e+\kappa_1)(1+e\kappa_2)\cos^2 p\pi}{(e+\kappa_1)^2 + (1+e\kappa_2)^2 + 2(e+\kappa_1)(1+e\kappa_2)\cos 2p\pi},$$

$$\mathbf{Q}(p) = \mathbf{I} + g(p)\mathbf{J},\ \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g(p) = i\beta \tan p\pi,\ \mathbf{J} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The exact analytical solution of equation (6) is obtained taking into account the features of the behavior of stresses and displacements near the tip of the interface discontinuity line and using the principle of analytical continuation, theorems of the Abelian type and Liouville and the method of Khrapkov factorization of the matrix coefficient $\mathbf{Q}(p)$ (Khrapkov, 1971) similarly to the solution of problems in works by Kaminskii et al. (1995, 1999, 2023). Having applied to the found solution the requirement of the stress finiteness at the end of the process zone, an equation was derived for calculating its relative length $x = l / L$ and the phase angle ψ of the stress in the zone:

$$x = C\sigma^2 \left\{ \frac{\cos^2(\omega \ln x + \varphi + \xi + \zeta)}{n^2} + \sin^2(\omega \ln x + \varphi + \xi + \zeta) \right\}, \tag{7}$$

$$\tan\psi = n \cdot \tan(\omega \ln x + \varphi + \xi + \zeta), \tag{8}$$

where

$$C = \pi \left[e' |N| \cdot |q'| G^+(-1)/q \right]^2,\ q = q_0^2 + q_1^2,\ q' = (q_0\tilde{q}_0 + q_1\tilde{q}_1) + i(q_1\tilde{q}_0 - q_0\tilde{q}_1),\ \zeta = \arg q,$$

$$q_0 = \tilde{Q}_{11}^{+}(-1),\ q_1 = \tilde{Q}_{12}^{+}(-1),\ \tilde{q}_0 = \tilde{Q}_{11}^{+}(-0,5-i\omega),\ \tilde{q}_1 = \tilde{Q}_{12}^{+}(-0,5-i\omega);$$

$$N = \frac{K^{+}(-0,5-i\omega)}{(0,5+i\omega)G^{+}(-0,5-i\omega)},\ \xi = \arg N;\ K^{\pm}(p) = \frac{\Gamma(1\mp p)}{\Gamma(0,5\mp p)},\ \exp\left[\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{\ln G(z)}{z-p}dz\right] = \begin{cases} G^{+}(p), \operatorname{Re} p<0, \\ G^{-}(p), \operatorname{Re} p>0; \end{cases}$$

$$\mathbf{Q}^{\pm}(p) = r^{\pm}(p)\left[\cosh\theta^{\pm}(p)\mathbf{I} + \sinh\theta^{\pm}(p)\mathbf{J}\right],\ r^{\pm}(p) = \left(1-\beta^2\right)^{1/4}\exp\left[\pm\frac{i}{4\pi}\int\limits_{-\infty}^{\infty}\frac{\ln H_1(t)}{t+ip}dt\right],\ H_1(t) = \frac{1-\beta^2\tanh^2\pi t}{1-\beta^2},$$

$$\theta^{\pm}(p) = \frac{\pm p}{2\pi}\int\limits_{-\infty}^{\infty}\frac{H_2(t)}{t+ip}dt,\ H_2(t) = \frac{\operatorname{arctanh}(\beta\cdot\tanh\pi t)}{t};\ \sigma = \frac{\left|K_I + iK_{II}\right|}{\tau_0\sqrt{L}},\ \varphi = \arctan(K_{II}/K_I);\ n = \sigma_0/\tau_0;$$

where σ and φ are the dimensionless modulus and phase angle of the external loading; Γ is the Eulers's gamma function.

The mode I and II energy release rates in the process zone G_1^{pz}, G_2^{pz} for the energy criterion of crack growth can be found as:

$$G_1^{pz} = \frac{8(1-\nu_1^2)L\tau_0^2}{E_1}\cdot\frac{e'^2\sigma^2|N|^2\cdot|q'M|}{Aq^2(1-\beta^2)^{1/4}h_1}[\cos(\omega\ln x+\varphi+\xi+\zeta)\cos(\omega\ln x+\varphi+\xi+\mu+\zeta_0+\theta_1) - \tag{9}$$

$$-\omega\sin(2\omega\ln x+2\varphi+2\xi+\zeta+\mu+\zeta_0+\theta_1)],$$

$$G_2^{pz} = \frac{8(1-\nu_1^2)L\tau_0^2}{E_1}\cdot\frac{e'^2\sigma^2|N|^2\cdot|q'M|}{Aq^2(1-\beta^2)^{1/4}h_1}[\sin(\omega\ln x+\varphi+\xi+\zeta)\sin(\omega\ln x+\varphi+\xi+\mu+\zeta_0+\theta_1) +$$

$$+\omega\sin(2\omega\ln x+2\varphi+2\xi+\zeta+\mu+\zeta_0+\theta_1)];$$

$$M = \frac{1-2i\omega}{3+2i\omega},\ \mu = \arg M;\ \ h_1 = \exp\left[\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\frac{\ln H_1(t)}{1+t^2}dt\right],\ \theta_1 = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{H_2(t)}{1+t^2}dt,\ \tilde{q} = \tilde{q}_0 - i\tilde{q}_1,\ \tilde{\zeta} = \arg\tilde{q}.$$

Adjoined with the corresponding components of the energy release rate, which correspond to the flow of elastic energy into the tip of the crack while neglecting the formation of the process zone and are determined independently, we calculate their full values for the I and II modes, $G_{1(2)} = G_{1(2)}^{el} + G_{1(2)}^{pz}$. G_1 i G_2 can be used in the corresponding criterion of the crack growth, i.e. $F\left(G_1/G_{1c}, G_2/G_{2c}\right) = 1$, where G_{1c}, G_{2c} are the fracture toughness of bonding material under the corresponding loading mode.

3. Numerical analysis of the model

Some results of numerical calculations of the process zone parameters according to formulas (7)-(9) are shown in Figs. 2-4. All calculations were performed for E_1/E_2=0,25, ν_1=ν_2=0,3.

According to the calculations, the length of the process zone increases nonlinearly with an increase in the external load specified by the dimensionless parameter σ (Fig. 2a). The dependence of the phase angle of the load in the zone on σ is less pronounced, but its slow growth with increasing load is also observed (Fig. 2b).

Fig. 3 exhibits the dependence of the process zone parameters on the phase angle of the external load φ. According to Fig. 3a, the length of the zone in this dependence has a minimum in the case of a more plastic bonding material (n>1) and a maximum in the case of a more brittle one (n<1). The phase angle of stresses in the zone ψ increases synchronously with the growth of the phase angle of the external load φ, remaining, however, smaller (Fig. 3b) than it is.

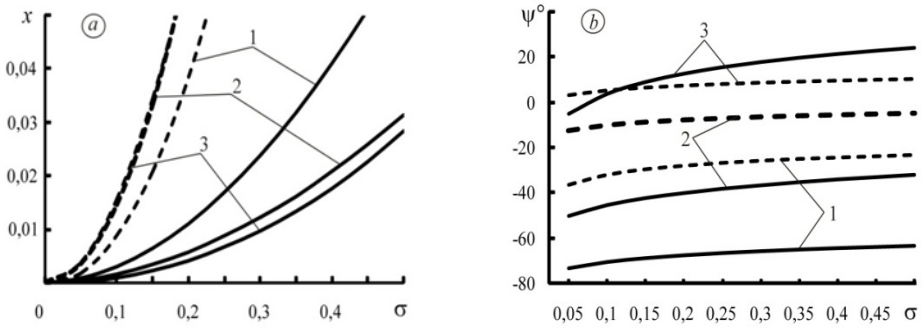


Fig. 2. The process zone parameters dependence on the dimensionless loading parameter σ. Dashed lines are for n=0,5; solid lines are for n=2. The phase angle of the external loading 1) $\varphi = -30°$; 2) $\varphi = 0°$; 3) $\varphi = 30°$.

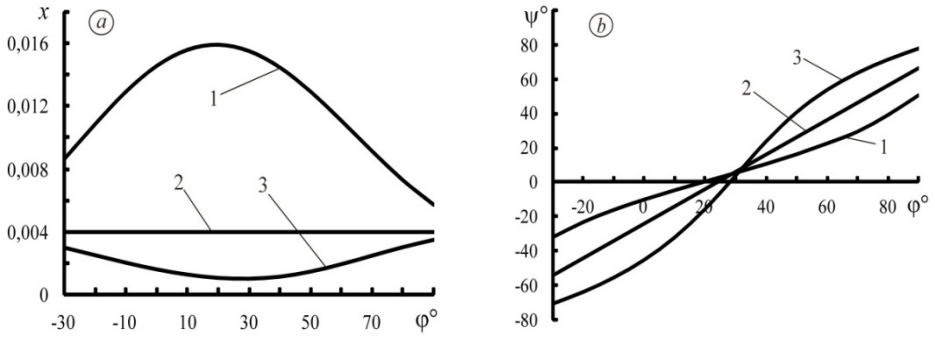


Fig. 3. The process zone parameters dependence on the external loading phase angle for σ=0,1: 1) n=0,5; 2) n=1; 3) n=2.

Fig. 4 shows the dependence of the normalized energy release rate in the process zone $G' = G^{pz}E_1 / 4(1-\nu_1^2)L\tau_0^2$ ($G^{pz} = G_1^{pz} + G_2^{pz}$) on the dimensionless loading parameter σ for partial cases of the Young's moduli ratio of the bonded materials E_1/E_2<1 and ν_1=ν_2=0.3. Cases of E_1/E_2>1 are not considered because they give identical results to E_1/E_2<1 after interchange of material characteristics. It is quite expected that G' increases as the load increases, acquiring the largest values for the identical materials ($E_1/E_2 \to 1$). At the same time, the value of the normalized rate of energy release in the process zone for a given load parameter σ does not depend neither on the phase angle of the load nor on the ratio of the bonding material limiting tensile and shear stresses.

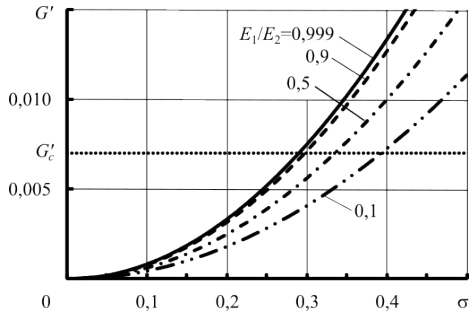


Fig. 4. Energy release rate vs. loading modulus.

4. Conclusions

The basic conditions laid down in the mathematical formulation of the model ensure obtaining an accurate analytical solution, which creates prospects for the practical use of the model due to its following features:

- the final calculation formulas for determining the length of the process zone, the phase angle of the load and the rate of energy release in the zone can be presented as a closed system of transcendental equations and ratios, the solution of which does not require the cumbersome calculations using "heavy" mathematical software packages;
- the use of the strength criterion of the Mises-Hill type allows the application of the model for the analysis of the conditions of propagation of interfacial cracks in bonding materials with a quasi-brittle mechanism of fracture;
- specifying the external load through the stress intensity factor near the tip of the crack makes the obtained solution general, not tied to a specific type of load. At the same time, this ensures the suitability of the model for studying the parameters of the process zone under conditions of mixed loading;
- material characteristics required to implement the model (tensile and shear strengths and fracture toughness) are available for most materials used as an adhesive in the relevant reference books, research papers or on the manufacturer's website.

Of course, the simplified nature of the proposed model of the process zone cannot ensure high accuracy of forecasting the parameters of the zone and the stress-strain state in the vicinity of the interface crack tip, but it enables their quick realistic assessment.

References

Barenblatt, G.I., 1962. The Mathematical Theory of Equilibrium Cracks in Brittle Fracture. Advances in Applied Mechanics 7, 55–129.

Dugdale, D.S., 1960. Yielding of steel sheets containing slits. Journal of the Mechanics and Physics of Solids 8, 100–104.

England, A.H., 1965. A crack between dissimilar media. Journal of Applied Mechanics 32(2), 400–402.

Erdogan, F., 1963. Stress distribution in a nonhomogeneous elastic plane with cracks. Journal of Applied Mechanics 30(2), 232–236.

Kaminskii, A.A., Kipnis, L.A., Kolmakova, V.A., 1995. Slip lines at the end of a cut at the interface of different media. International Applied Mechanics 31(6), 491–495.

Kaminskii, A.A., Kipnis, L.A., Kolmakova, V.A., 1999. On the Dugdale model for a crack at the interface of different media. International Applied Mechanics 35(1), 58–63.

Kaminsky, A., Dudyk, M., Reshitnyk, Y., Chornoivan, Y., 2023. An analytical method of modeling the process zone near the tip of an interface crack due to its kinking from the interface of quasi-elastic materials. International Journal of Solids and Structures 267, 112117.

Khrapkov, A., 1971. Certain cases of the elastic equilibrium of an infinite wedge with a nonsymmetric notch at the vertex, subjected to concentrated forces. Journal of Applied Mathematics and Mechanics 35(4), 625–637.

Leonov, M.Y., Panasyuk, V.V., 1959. Growth of the smallest cracks in solids. Prikladna Mechanika 5(4), 391–401.

Needleman, A., 1987. A continuum model for void nucleation by inclusion debonding. Journal of Applied Mechanics 54(3), 525–531.

Rice, J.R., Sih, G.C., 1965. Plane problems of cracks in dissimilar media. Journal of Applied Mechanics 32(2), 418–423.