DOI 10.1007/s10958-022-06065-7

Journal of Mathematical Sciences, Vol. 265, No. 3, August, 2022

COHESIVE MODEL OF THE PROCESS ZONE NEAR THE TIP OF A CRACK ORIGINATING FROM A CORNER POINT OF THE BROKEN INTERFACE

A. O. Kamins'kyi,[1] **M. V. Dudyk,**[2] **and Yu. V. Reshitnyk**[3]

UDC 539.375

Within the framework of a cohesive model, we determine the parameters of the process zone in the binding material at the tip of an interface crack originating from the corner point of the broken interface of two different homogeneous isotropic materials. The indicated zone is modeled by the line of discontinuity of displacements where the stresses satisfy the Mises–Hill fracture criterion. By the Mellin integral transformation, we reduce the problem to the Wiener–Hopf vector equation whose analytic solution is found by the method of successive approximations. We deduce the equations for the length of the process zone and the phase length of stresses in this zone, as well as the expression for the crack-tip opening displacement. Finally, we perform the numerical analysis of parameters of the process zone.

Keywords: interface crack, process zone, cohesive model.

Introduction

The process of crack propagation on the interface of joined materials is one of the most probable causes of fractures of the composites. Crack initiation is preceded by the formation of a process zone with nonlinear dependence of the level of strains on stresses. This zone is located on the continuation of a crack in the binding material. Note that, in fracture mechanics, this phenomenon is described by an extensively developed cohesive model [12, 14, 15, 17–20]. According to this model, the process zone is represented in the form of a surface of discontinuity of displacements with given phenomenological relations between the forces of attraction of the crack faces and the distance between the faces specified on this surface. The analyzed model is a formal analog of the Leonov–Panasyuk–Dugdale model [9, 10].

The nonlinear character of the determining equations in the process zone explains the necessity of investigation of cohesive models mainly by using numerical methods. However, in the case of small-scale process zones, it might be possible to perform analytic evaluation of their parameters with the help of the Wiener–Hopf method [8]. In this case, external loads are taken into account by means of asymptotic matching of the solutions in the formulation of the conditions at infinity. Various specific features of realization of this method under the conditions of plane deformation are demonstrated by the evaluation of the parameters of interface and lateral small-scale process zones in the vicinities of interface cracks formed in piecewise-homogeneous bodies within the framework of the Leonov–Panasyuk–Dugdale model in which the transition of the material into the prefracture state is determined by the Tresca–Saint-Venant condition [3–6].

By using the cohesive model and the Mises–Hill fracture condition for stresses, the exact analytic solution

[1] S. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kyiv, Ukraine.

[2] P. Tychyna Uman State Pedagogical University, Uman, Ukraine; e-mail: dudik_m@hotmail.com.

[3] P. Tychyna Uman State Pedagogical University, Uman, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 62, No. 4, pp. 112–123, October–December, 2019. Original article submitted August 21, 2019.

 1072-3374/22/2653–0474 © 2022 Springer Nature Switzerland AG

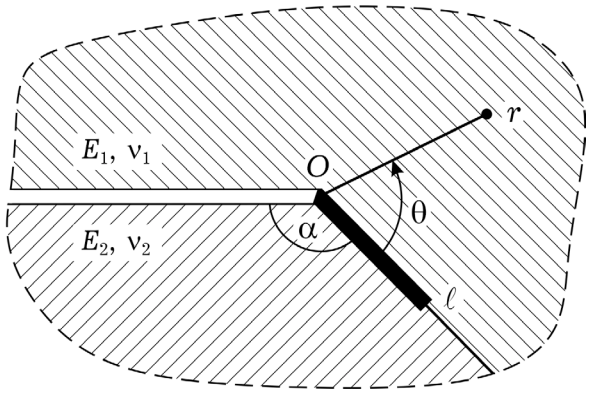


Fig. 1. Schematic diagram of the problem.

of the problem of formation of a small-scale process zone in the binding material near the tip of a crack located on the plane interface of two different homogeneous isotropic materials was obtained in [2].

In the present work, we consider a similar problem for an interface crack originating from a corner point of the broken interface of materials. By using the Mellin integral transformation, we reduce the problem to the Wiener–Hopf vector equation, which is solved by the method of successive approximations. As the basis approximation, we use the solution of the problem with plane interface for a partial case studied in [2]. We deduce equations for the length of the process zone, for the phase angle of stresses in this zone, and for the crack-tip opening displacements.

1. Statement of the Problem

Under the conditions of plane deformation, we consider the problem of evaluation of the parameters of a small-scale process zone formed in the binding material near the tip of an interface crack originating from a corner point of the broken interface of two elastic homogeneous isotropic materials with Young's moduli E_1 and E_2 and Poisson's ratios ν_1 and ν_2, respectively. In view of the anisotropy of the characteristics of binding material in the process zone typical of the cohesive models, by neglecting the thickness of the interlayer of binding material, we simulate the process zone by the line of discontinuity of displacements. On this line, the normal and tangential stresses satisfy the Mises–Hill criterion

$$\left(\frac{\sigma_\theta}{\sigma_0}\right)^2 + \left(\frac{\tau_{r\theta}}{\tau_0}\right)^2 = 1, \tag{1}$$

where σ_0 and τ_0 are the tensile strength and the shear resistances of the binding material experimentally determined as the mean values of the normal and tangential stresses in the process zone for the corresponding loading mode.

Assume that the length of the small-scale process zone ℓ is much smaller than the crack length L and the other typical sizes of the body. If we study the stress-strain state in a vicinity of this zone, then it is possible to reduce the original problem to the problem of analysis of a discontinuity line of finite length propagating from the tip of a semiinfinite interface crack in a piecewise homogeneous plane along the interface of two different elastic materials (Fig. 1).

We formulate the condition at infinity as the requirement of joining of the required solution at distances $\ell \ll r \ll L$ with the known solution of a similar problem of the theory of elasticity in the absence of discontinuity lines obtained in [3, 4].

Suppose that the crack faces are free of stresses. Thus, in view of condition (1), we arrive at the static boundary-value problem of the theory of elasticity with the following boundary conditions:

$$\theta = -\alpha \cup 2\pi - \alpha: \quad \sigma_\theta = \tau_{r\theta} = 0, \tag{2}$$

$$\theta = 0: \quad \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0, \tag{3}$$

$$\theta = 0, \quad r < \ell: \quad \sigma_\theta = \sigma_0 \cos\psi(r), \quad \tau_{r\theta} = \tau_0 \sin\psi(r),$$

$$\theta = 0, \quad r > \ell: \quad \langle u_\theta \rangle = \langle u_r \rangle = 0, \tag{4}$$

$$\theta = 0, \quad r \to \infty: \quad \sigma_\theta \sim \sum_i C_i F_\sigma(\alpha, \lambda_i) r^{\lambda_i}, \quad \tau_{r\theta} \sim \sum_i C_i F_\tau(\alpha, \lambda_i) r^{\lambda_i}. \tag{5}$$

Here, $\langle f \rangle$ is a jump of the quantity f on the interface; $\psi(r)$ is the phase angle of stresses in the process zone (in what follows, we assume that it is constant and equal to the mean value ψ); $F_\tau(\alpha, \lambda)$ and $F_\sigma(\alpha, \lambda)$ are known functions from the works [3, 4], respectively; C_i are arbitrary constants characterizing the intensity of external loading and regarded as known (by condition), and the exponents of the singularities of stresses in the vicinity of the crack tip λ_i, $-1 < \operatorname{Re}\lambda_i < 0$, are the roots of the characteristic equation for a similar problem in the absence of the process zone [3, 4]:

$$D(\lambda) = 0,$$

where

$$D(\lambda) = -(1+\kappa_1)^2 t_1 - 4(1+\kappa_1)(e-1)t_1 t_2 - e^2(1+\kappa_2)^2 t_3$$

$$+ 4(e-1)^2 t_1 t_3 + 4e(1+\kappa_2)(e-1)t_3 t_4 + 2e(1+\kappa_2)(1+\kappa_1)t_5,$$

$$t_1 = (\lambda+1)^2 \sin^2\alpha - \sin^2(\lambda+1)\alpha, \quad t_2 = \sin^2(\lambda+1)(2\pi-\alpha),$$

$$t_3 = (\lambda+1)^2 \sin^2\alpha - \sin^2(\lambda+1)(2\pi-\alpha),$$

$$t_4 = \sin^2(\lambda+1)\alpha, \quad t_5 = t_1 + \sin(\lambda+1)\alpha \sin 2\lambda\pi \cos(\lambda+1)(2\pi-\alpha),$$

$$e = \frac{1+\nu_2}{1+\nu_1} \cdot \frac{E_1}{E_2}, \quad \kappa_{1(2)} = 3 - 4\nu_{1(2)}.$$

At the end of the process zone, we observe the realization of asymptotics corresponding to the singular part

of solution of the homogeneous boundary-value problem for a semiinfinite line of discontinuity of displacements on the rectilinear interface of two different elastic materials, which is similar to an interface crack. In particular, we have the following asymptotics for stresses [16]:

$$r \to \ell + 0, \quad \sigma_\theta(r,0) + i\tau_{r\theta}(r,0) \sim \frac{k}{\sqrt{2\pi}}(r-\ell)^{-0.5+i\omega}, \tag{6}$$

where

$$\omega = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta}, \quad \beta = \frac{(1+e\kappa_2)-(e+\kappa_1)}{(1+e\kappa_2)+(e+\kappa_1)}$$

is the Dundurs parameter, and $k = k_1 + ik_2$ is the local stress intensity factor (SIF) at the end of the zone.

We seek the solution of the posed boundary-value problem in the form of a sum of solutions of two problems. The first problem differs from the problem formulated above by the replacement of the first pair of conditions in (4) by the following conditions:

$$\theta = 0, \quad r < \ell: \quad \sigma_\theta = \sigma_0 \cos\psi - \sum_i C_i F_\sigma(\alpha,\lambda_i) r^{\lambda_i},$$
$$\tau_{r\theta} = \tau_0 \sin\psi - \sum_i C_i F_\tau(\alpha,\lambda_i) r^{\lambda_i}. \tag{7}$$

According to (5), this results in the decay of stresses at infinity as $o(1/r)$. The role of the second problem is played by a similar problem without discontinuity lines whose solution is known [3, 4]. Hence, our task is reduced to the solution of the first problem.

2. Solution of the Problem

By applying the Mellin integral transformation to the equilibrium equations, condition of compatibility of deformations, Hooke's law, boundary conditions (2) and (3), the second pair of conditions in (4), and conditions (7), we arrive at the Wiener–Hopf vector equation for the first problem in a strip containing the imaginary axis:

$$\mathbf{\Phi}^+(p) + \mathbf{F}(p) = \mathbf{G}(p)\mathbf{\Phi}^-(p), \quad -\varepsilon_1 < \operatorname{Re} p < \varepsilon_2, \tag{8}$$

where

$$\mathbf{\Phi}^+(p) = \int_1^\infty \begin{pmatrix} \sigma_\theta(\rho\ell,0) \\ \tau_{r\theta}(\rho\ell,0) \end{pmatrix} \rho^p \, d\rho, \quad \mathbf{F}(p) = \begin{pmatrix} F_1(p) \\ F_2(p) \end{pmatrix},$$

$$F_1(p) = \frac{\sigma_0 \cos\psi}{p+1} - \sum_i \frac{C_i F_\sigma(\alpha,\lambda_i)\ell^{\lambda_i}}{p+1+\lambda_i},$$

$$F_2(p) = \frac{\tau_0 \sin\psi}{p+1} - \sum_i \frac{C_i F_\tau(\alpha,\lambda_i)\ell^{\lambda_i}}{p+1+\lambda_i},$$

$$\mathbf{\Phi}^-(p) = \frac{E_1}{4(1-\nu_1^2)} \int_0^1 \left\langle \frac{\partial}{\partial r} \begin{pmatrix} u_\theta \\ u_r \end{pmatrix} \right\rangle \Bigg|_{\substack{\theta=0 \\ r=\rho\ell}} \rho^p \, d\rho,$$

$$\mathbf{G}(p) = -\frac{1+\kappa_1}{D(-1-p)} \begin{pmatrix} g_{11}(p) & 2g_{12}(p) \\ 2g_{21}(p) & g_{22}(p) \end{pmatrix},$$

$$g_{11}(p) = e(1+\kappa_2)d_1(p)\,d_6(p) - (1+\kappa_1)d_2(p)\,d_4(p),$$

$$g_{12}(p) = 2(1-e)\,d_1(p)\,d_2(p) + e(1+\kappa_2)\,d_1(p)\,d_8(p) - (1+\kappa_1)\,d_2(p)\,d_7(p),$$

$$g_{21}(p) = e(1+\kappa_2)\,d_1(p)\,d_9(p) - 2(1-e)\,d_1(p)\,d_2(p) - (1+\kappa_1)\,d_2(p)\,d_5(p),$$

$$g_{22}(p) = (1+\kappa_1)\,d_2(p)\,d_3(p) - e(1+\kappa_2)\,d_1(p)\,d_{10}(p),$$

$$d_1(p) = p^2\sin^2\alpha - \sin^2 p(2\pi-\alpha), \quad d_2(p) = p^2\sin^2\alpha - \sin^2 p\alpha,$$

$$d_3(p) = p\sin 2\alpha - \sin 2p(2\pi-\alpha), \quad d_4(p) = p\sin 2\alpha + \sin 2p(2\pi-\alpha),$$

$$d_5(p) = p\sin^2\alpha + \sin^2 p(2\pi-\alpha), \quad d_6(p) = p\sin 2\alpha - \sin 2p\alpha,$$

$$d_7(p) = p\sin^2\alpha - \sin^2 p(2\pi-\alpha), \quad d_8(p) = p\sin^2\alpha - \sin^2 p\alpha,$$

$$d_9(p) = p\sin^2\alpha + \sin^2 p\alpha, \quad d_{10}(p) = p\sin 2\alpha + \sin 2p\alpha$$

(ε_1 and ε_2 are sufficiently small positive numbers).

The solution of Eq. (8) includes the factorization of the matrix $\mathbf{G}(p)$, i.e., its representation in the form of the product of two matrices $\mathbf{G}^+(p)$ and $\mathbf{G}^-(p)$ analytic in the half planes $\operatorname{Re} p < \varepsilon_2$ and $\operatorname{Re} p > -\varepsilon_1$, respectively. However, the matrix $\mathbf{G}(p)$ cannot be attributed to any known type admitting the exact factorization in the closed analytic form [21]. Therefore, it is assumed that this matrix can be decomposed into two matrices such that one of them admits the exact factorization, while the second one is supposed to be small as compared with the first matrix in the domain of validity of Eq. (8):

$$\mathbf{G}(p) = \mathbf{G}_0(p) + \mathbf{G}'(p), \quad |\mathbf{G}_0(p)| \gg |\mathbf{G}'(p)|, \quad -\varepsilon_1 < \operatorname{Re} p < \varepsilon_2. \tag{9}$$

As $\mathbf{G}_0(p)$, we choose the matrix $\mathbf{G}(p)$ obtained for the plane interface [2]:

$$\mathbf{G}_0(p) = \mathbf{G}(p)\big|_{\alpha=\pi} = -A\cdot\tan p\pi G(p)\mathbf{Q}(p), \tag{10}$$

where

$$G(p) = \frac{4(e+\kappa_1)(1+e\kappa_2)\cos^2 p\pi}{(e+\kappa_1)^2+(1+e\kappa_2)^2+2(e+\kappa_1)(1+e\kappa_2)\cos 2p\pi},$$

$$A = \frac{(1+\kappa_1)(1+\kappa_1+e(1+\kappa_2))}{2(e+\kappa_1)(1+e\kappa_2)}, \quad \mathbf{Q}(p) = \mathbf{I} + g(p)\mathbf{J}, \quad \mathbf{J} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Here, $\mathbf{I}$ is the identity matrix and $g(p)=i\beta\tan p\pi$. The factorization of the matrix $\mathbf{Q}(p)$ is carried out on the imaginary axis according to the Khrapkov formulas [13]:

$$\begin{gathered}\mathbf{Q}(p) = \mathbf{Q}^+(p)\mathbf{Q}^-(p), \quad \operatorname{Re} p = 0, \\ \mathbf{Q}^\pm(p) = r^\pm(p)[\cosh\theta^\pm(p)\mathbf{I}+\sinh\theta^\pm(p)\mathbf{J}],\end{gathered} \tag{11}$$

where

$$r^\pm(p) = (1-\beta^2)^{1/4}\exp\left[\pm\frac{i}{4\pi}\int\limits_{-\infty}^{\infty}\frac{\ln H_1(t)}{t+ip}dt\right], \quad H_1(t) = \frac{1-\beta^2\tanh^2\pi t}{1-\beta^2},$$

$$\theta^\pm(p) = \frac{\pm p}{2\pi}\int\limits_{-\infty}^{\infty}\frac{H_2(t)}{t+ip}dt, \quad H_2(t) = \frac{\operatorname{artanh}(\beta\cdot\tanh\pi t)}{t}.$$

Here, $\mathbf{Q}^+(p)$ and $\mathbf{Q}^-(p)$ are matrices analytic in the half planes $\operatorname{Re}p<0$ and $\operatorname{Re}p>0$, respectively. The scalar coefficients of the matrix $\mathbf{G}_0(p)$ can be factorized as follows [1, 8]:

$$\begin{gathered}G(p) = \frac{G^+(p)}{G^-(p)}, \quad \operatorname{Re}p = 0, \\ \exp\left[\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{\ln G(z)}{z-p}dz\right] = \begin{cases} G^+(p), & \operatorname{Re} p<0, \\ G^-(p), & \operatorname{Re} p>0, \end{cases}\end{gathered} \tag{12}$$

$$\tan p\pi = \frac{p}{\mathcal{K}^+(p)\mathcal{K}^-(p)}, \quad \mathcal{K}^\pm(p) = \frac{\Gamma(1\mp p)}{\Gamma(0.5\mp p)}$$

($\Gamma(z)$ is the Euler gamma-function). Thus, we finally get the following factorization of the matrix $\mathbf{G}_0(p)$:

$$\mathbf{G}_0(p) = \mathbf{G}_0^+(p)\mathbf{G}_0^-(p), \quad \operatorname{Re} p = 0, \tag{13}$$

where

$$\mathbf{G}_0^+(p) = \frac{pG^+(p)}{\mathcal{K}^+(p)}\mathbf{Q}^+(p), \quad \operatorname{Re} p < 0,$$

$$\mathbf{G}_0^-(p) = \frac{-A}{G^-(p)\mathcal{K}^-(p)}\mathbf{Q}^-(p), \quad \operatorname{Re} p < 0.$$

By virtue of the assumptions made above, we can represent the solution of Eq. (8) in the form of expansion in powers of the matrix perturbation $\mathbf{G}'(p)$:

$$\mathbf{\Phi}^\pm(p) = \mathbf{\Phi}_0^\pm(p) + \mathbf{\Phi}_1^\pm(p) + \mathbf{\Phi}_2^\pm(p) + \ldots. \tag{14}$$

Thus, in view of relations (9)–(14), in the zero-order approximation, Eq. (8) is reduced to the equation

$$[\mathbf{G}_0^+(p)]^{-1}\big(\mathbf{\Phi}_0^+(p)+\mathbf{F}_0(p)\big) = \mathbf{G}_0^-(p)\mathbf{\Phi}_0^-(p), \quad \operatorname{Re} p = 0, \tag{15}$$

where the exponent -1 denotes the inverse matrix. In view of the asymptotic relations (6), the solution of Eq. (15) obtained in [15] by using the Wiener–Hopf method [8] takes the form

$$\begin{aligned} \mathbf{\Phi}_0^+(p) &= -\mathbf{G}_0^+(p)\mathbf{F}_0^+(p), \quad \operatorname{Re} p < 0, \\ \mathbf{\Phi}_0^-(p) &= -[\mathbf{G}_0^-(p)]^{-1}\mathbf{F}_0^-(p), \quad \operatorname{Re} p > 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbf{F}_0^+(p) = {} & \frac{1}{p+1}\Big([\mathbf{G}_0^+(p)]^{-1} - [\mathbf{G}_0^+(-1)]^{-1}\Big)\begin{pmatrix}\sigma_0\cos\psi\\ \tau_0\sin\psi\end{pmatrix} \\ & - \sum_i \frac{C_i\ell^{\lambda_i}}{p+1+\lambda_i}\Big([\mathbf{G}_0^+(p)]^{-1} - [\mathbf{G}_0^+(-1-\lambda_i)]^{-1}\Big)\begin{pmatrix}F_\sigma(\alpha,\lambda_i)\\ F_\tau(\alpha,\lambda_i)\end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{F}_0^-(p) = {} & -\frac{[\mathbf{G}_0^+(-1)]^{-1}}{p+1}\begin{pmatrix}\sigma_0\cos\psi\\ \tau_0\sin\psi\end{pmatrix} \\ & + \sum_i \frac{C_i\ell^{\lambda_i}}{p+1+\lambda_i}[\mathbf{G}_0^+(-1-\lambda_i)]^{-1}\begin{pmatrix}F_\sigma(\alpha,\lambda_i)\\ F_\tau(\alpha,\lambda_i)\end{pmatrix}. \end{aligned}$$

For the corrections $\mathbf{\Phi}_1^{\pm}(p)$ in the first approximation, by using Eq. (8), we obtain

$$\mathbf{\Phi}_1^{+}(p) = \mathbf{G}_0(p)\,\mathbf{\Phi}_1^{-}(p) + \mathbf{G}'(p)\,\mathbf{\Phi}_0^{-}(p), \qquad \operatorname{Re} p = 0,$$

where the matrix perturbation $\mathbf{G}'(p)$ is determined, according to (9), as the difference of the matrices $\mathbf{G}(p) - \mathbf{G}_0(p)$. We now introduce a vector function $\mathbf{F}_1(p) = -\mathbf{G}'(p)\,\mathbf{\Phi}_0^{-}(p)$. Thus, with regard for the factorization (13) of the matrix $\mathbf{G}_0(p)$, we arrive at the equation

$$\left[\mathbf{G}_0^{+}(p)\right]^{-1}\left(\mathbf{\Phi}_1^{+}(p)+\mathbf{F}_1(p)\right) = \mathbf{G}_0^{-}(p)\,\mathbf{\Phi}_1^{-}(p), \quad \operatorname{Re} p = 0, \tag{17}$$

by analogy with (15). Replacing $\left[\mathbf{G}_0^{+}(p)\right]^{-1}\mathbf{F}_1(p)$ by the difference of the boundary values of the analytic vectors

$$\left[\mathbf{G}_0^{+}(p)\right]^{-1}\mathbf{F}_1(p) = \mathbf{F}_1^{+}(p) - \mathbf{F}_1^{-}(p), \tag{18}$$

where

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left[\mathbf{G}_0^{+}(z)\right]^{-1}\mathbf{F}_1(z)}{z-p}\,dz = \begin{cases}\mathbf{F}_1^{+}(p), & \operatorname{Re} p<0,\\ \mathbf{F}_1^{-}(p), & \operatorname{Re} p>0,\end{cases}$$

we solve Eq. (17) by analogy with Eq. (15) according the Wiener–Hopf method [2]. As a result, we get the first-order (in perturbation) correction to solution (16):

$$\begin{aligned}\mathbf{\Phi}_1^{+}(p) &= -\mathbf{G}_0^{+}(p)\,\mathbf{F}_1^{+}(p), \quad \operatorname{Re} p < 0,\\ \mathbf{\Phi}_1^{-}(p) &= -\left[\mathbf{G}_0^{-}(p)\right]^{-1}\mathbf{F}_1^{-}(p), \quad \operatorname{Re} p > 0.\end{aligned} \tag{19}$$

The next approximations are obtained from (17)–(19) by successive replacements

$$\mathbf{\Phi}_{i-1}^{\pm}(p) \to \mathbf{\Phi}_i^{\pm}(p) \quad \text{and} \quad \mathbf{F}_1(p) \to \mathbf{F}_i(p) = -\mathbf{G}'(p)\,\mathbf{\Phi}_{i-1}^{-}(p), \ i = 2, 3, \ldots.$$

As a result, we arrive at the general solution of Eq. (8):

$$\begin{aligned}\mathbf{\Phi}^{+}(p) &= -\mathbf{G}_0^{+}(p)\sum_{n=0}^{\infty}\mathbf{F}_n^{+}(p), \quad \operatorname{Re} p < 0,\\ \mathbf{\Phi}^{-}(p) &= -\left[\mathbf{G}_0^{-}(p)\right]^{-1}\sum_{n=0}^{\infty}\mathbf{F}_n^{-}(p), \quad \operatorname{Re} p > 0,\end{aligned} \tag{20}$$

where

$$\mathbf{F}_n^{\pm}(p) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{[\mathbf{G}_0^{+}(z)]^{-1}\mathbf{G}'(z)[\mathbf{G}_0^{-}(z)]^{-1}\mathbf{F}_{n-1}^{-}(z)}{z-p}dz.$$

The condition of correctness of solution (20) is reduced to the condition of convergence of its series, i.e., to the validity of the inequalities $\left|\mathbf{F}_n^{\pm}(p)\right| < \left|\mathbf{F}_{n-1}^{\pm}(p)\right|$, which are equivalent to the following condition:

$$\left|[\mathbf{G}_0^{+}(p)]^{-1}\mathbf{G}'(p)[\mathbf{G}_0^{-}(p)]^{-1}\right| < 1. \tag{21}$$

3. Determination of the Parameters of the Process Zone

Solution (20) yields the following asymptotics:

$$\mathbf{\Phi}^{+}(p) \sim -\frac{1}{\sqrt{-p}}\mathbf{Q}^{+}(p)\left\{[\mathbf{G}_0^{+}(-1)]^{-1}\begin{pmatrix}\sigma_0\cos\psi\\ \tau_0\sin\psi\end{pmatrix}\right.$$

$$-\sum_i C_i\ell^{\lambda_i}[\mathbf{G}_0^{+}(-1-\lambda_i)]^{-1}\begin{pmatrix}F_\sigma(\alpha,\lambda_i)\\ F_\tau(\alpha,\lambda_i)\end{pmatrix}$$

$$\left.+\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}[\mathbf{G}_0^{+}(z)]^{-1}\sum_{n=0}^{\infty}\mathbf{F}_n^{+}(z)\,dz\right\}, \qquad p\to\infty.$$

On the other hand, by the Abel-type theorem [8], it follows from relations (6) that

$$\Phi_1^{+}(p) + i\Phi_2^{+}(p) \sim \frac{k}{\sqrt{-2p\ell}}\Gamma(0.5+i\omega)(-p)^{-i\omega}, \quad p\to\infty.$$

Comparing these asymptotics, in view of the boundedness of stresses at the end of the process zone, which yields the condition $k=0$, we arrive at the vector equation

$$[\mathbf{G}_0^{+}(-1)]^{-1}\begin{pmatrix}\sigma_0\cos\psi\\ \tau_0\sin\psi\end{pmatrix} - \sum_i C_i\ell^{\lambda_i}[\mathbf{G}_0^{+}(-1-\lambda_i)]^{-1}\begin{pmatrix}F_\sigma(\alpha,\lambda_i)\\ F_\tau(\alpha,\lambda_i)\end{pmatrix}$$

$$+\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}[\mathbf{G}_0^{+}(z)]^{-1}\sum_{n=1}^{\infty}\mathbf{F}_n(z)\,dz = 0.$$

Passing to the scalar representation, we arrive at a transcendental equation for the length of the process zone

$$\left(\sum_i \frac{C_i \ell^{\lambda_i} \mathcal{K}^+(-1-\lambda_i)}{\sigma_0(1+\lambda_i)G^+(-1-\lambda_i)} X_1(\lambda_i)\right)^2$$

$$+ \left(\sum_i \frac{C_i \ell^{\lambda_i} \mathcal{K}^+(-1-\lambda_i)}{\tau_0(1+\lambda_i)G^+(-1-\lambda_i)} X_2(\lambda_i)\right)^2 = 1 .$$

Further, we can find the phase angle of stresses in this zone by the formula

$$\psi = \arctan\left[\frac{\sigma_0 \sum_i \frac{C_i \ell^{\lambda_i} \mathcal{K}^+(-1-\lambda_i)}{(1+\lambda_i)G^+(-1-\lambda_i)} X_2(\lambda_i)}{\tau_0 \sum_i \frac{C_i \ell^{\lambda_i} \mathcal{K}^+(-1-\lambda_i)}{(1+\lambda_i)G^+(-1-\lambda_i)} X_1(\lambda_i)}\right].$$

Here,

$$\begin{pmatrix} X_1(\lambda) \\ X_2(\lambda) \end{pmatrix} = \mathbf{M}_0^{-1}\left\{\mathbf{I} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{[\mathbf{G}_0^+(z)]^{-1}\mathbf{G}'(z)[\mathbf{G}_0^-(z)]^{-1}}{z+1+\lambda} dz\right\}$$

$$\times [\mathbf{Q}^+(-1-\lambda)]^{-1}\begin{pmatrix} F_\sigma(\alpha,\lambda) \\ F_\tau(\alpha,\lambda) \end{pmatrix},$$

$$\mathbf{M}_0 = \frac{\mathcal{K}^+(-1)}{G^+(-1)}\left\{\mathbf{I} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{[\mathbf{G}_0^+(z)]^{-1}\mathbf{G}'(z)[\mathbf{G}_0^-(z)]^{-1}}{z+1} dz\right\}[\mathbf{Q}^+(-1)]^{-1}.$$

By using the definition of the vector $\mathbf{\Phi}^-(p)$ in (8), we obtain the components of the jump of displacements at the crack tip in the following form:

$$\langle \mathbf{u}(0,0)\rangle = -\frac{4\left(1-\nu_1^2\right)}{E_1}\mathbf{\Phi}^-(0) = \frac{4\left(1-\nu_1^2\right)}{E_1}[\mathbf{G}_0^-(0)]^{-1}\sum_{n=0}^{\infty}\mathbf{F}_n^-(0).$$

This formula enables us to find the jumps of normal and tangential displacements in the process zone, which can be used to determine the ultimate load according to the following condition accepted in the model of cohesive zone [14, 20]:

$$\left(\frac{\langle u_\theta(0,0)\rangle}{\delta_n}\right)^2 + \left(\frac{\langle u_r(0,0)\rangle}{\delta_\tau}\right)^2 = 1.$$

Here, δ_n and δ_τ are, respectively, the critical values of the quantities whose attainment leads to the onset of crack propagation in the cases of pure tensile or shear strains formed in this zone.

4. Analysis of the Results

The numerical realization of the obtained solution of the problem with regard for the high-order approximations is complicated by increase in the multiplicity of Cauchy-type integrals in (20). Therefore, it is necessary to restrict ourselves to a relatively small number of terms used in calculations by replacing condition (21) with the following severer condition for the matrix perturbation:

$$\left|[\mathbf{G}_0^+(p)]^{-1}\mathbf{G}'(p)[\mathbf{G}_0^-(p)]^{-1}\right| \ll 1.$$

In this connection, we expect to get the highest efficiency of approximate solution of the problem of determination of the small-scale process zone for the kink angles of the interface of materials α that are close to $\alpha_0 = 180^\circ$. This angle corresponds to the zero-order approximation (10) for the matrix coefficient of the Wiener–Hopf equation (8). As shown in [3], the stress-strain state formed near the interface crack tip for the indicated kink angles is determined by two complex-conjugate singularity exponents $\lambda_1 = \overline{\lambda}_2 = \lambda_r + i\lambda_m$ and by the complex stress intensity factor $K = K_1 + iK_2$ appearing in the coefficients of expansions in (5):

$$C_1 = \overline{C}_2 = \frac{KL^{-i\lambda_m}}{\sqrt{2\pi}}.$$

Therefore, if we consider only small deviations of the shape of interface from the plane surface, then, as in [3], we specify the external load by the dimensionless parameter

$$\sigma = \frac{|K|\,L^{\lambda_r}}{\sqrt{2\pi}\,\sigma_0}$$

and the phase angle of a load

$$\varphi = \arctan\frac{K_2}{K_1}.$$

The results of numerical evaluation of the parameters of the process zone in the first approximation are shown in Figs. 2–5. The relative length

$$x = \frac{\ell}{L}$$

is presented in Figs. 2a–5a, the phase angle ψ of stresses in the process zone is depicted in Figs. 2b–5b, and the crack opening displacement

$$\delta u' = \frac{\delta u_1(0,0)}{L}\,\frac{E_1}{4(1-\nu_1^2)}$$

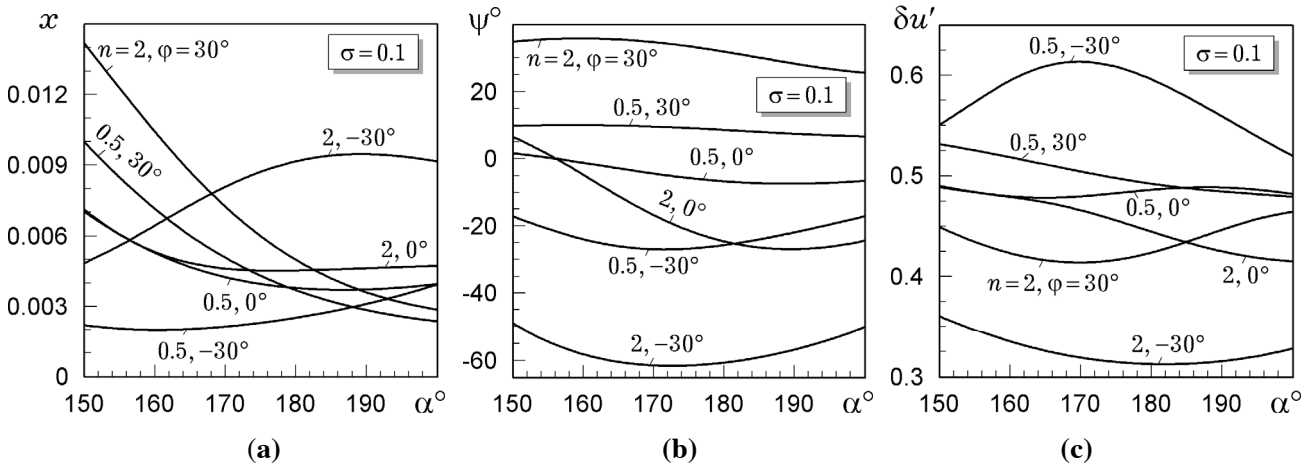


Fig. 2. Dependences of the parameters of the process zone on the kink angle α.

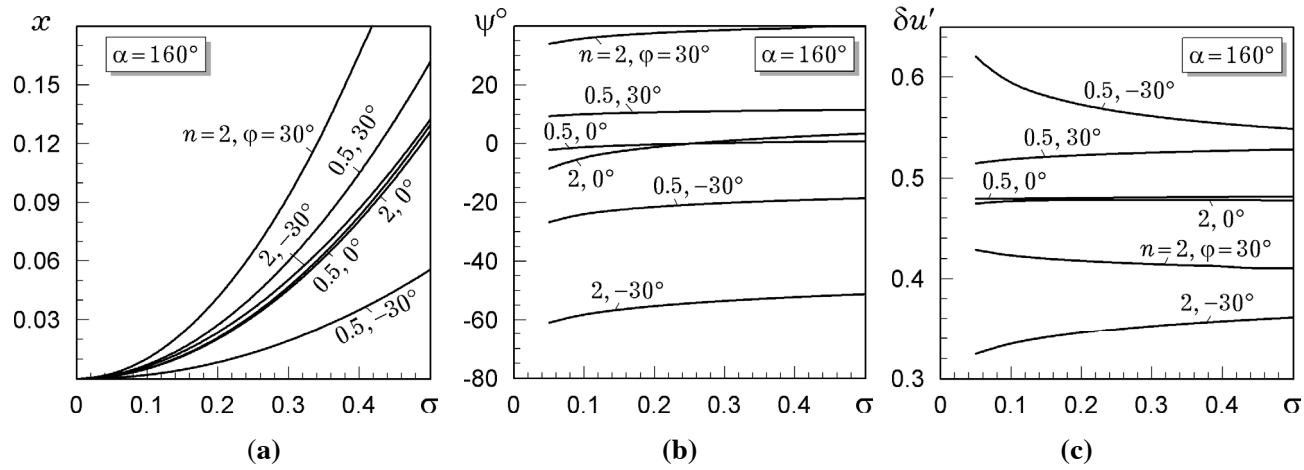


Fig. 3. Dependences of the parameters of the process zone on the dimensionless parameter of loading σ.

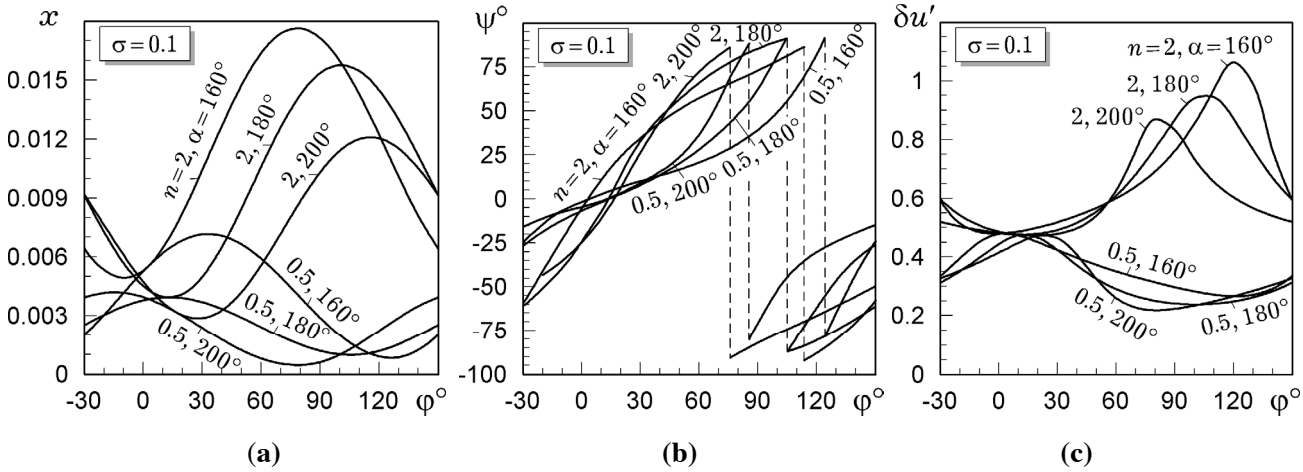


Fig. 4. Dependences of the parameters of the process zone on the phase angle φ of external loading.

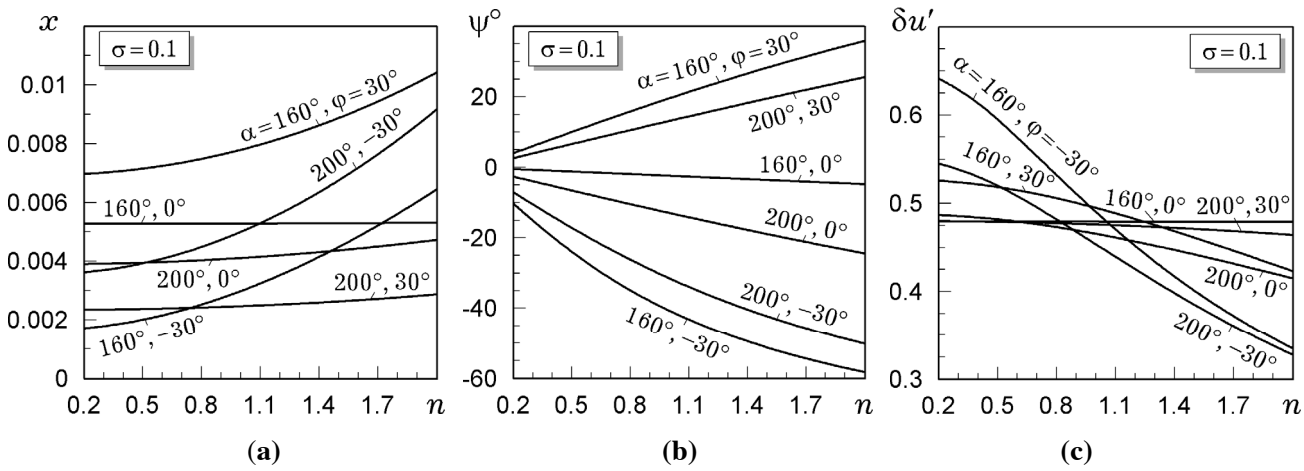


Fig. 5. Dependences of the parameters of the process zone on the ratio n of tensile and shear strengths of the binding material.

in displayed Figs. 2c–5c. All calculations were carried out for $E_1/E_2 = 0.5$, $\nu_1 = \nu_2 = 0.3$. We denote the ratio of the tensile and shear strengths of the binding material by

$$n = \frac{\sigma_0}{\tau_0}.$$

According to Fig. 2, the parameters of the process zone depend on the kink angle of the interface of materials α and deviate from the values corresponding to the plane boundary both to the side of higher values and to the side of lower values.

The length of the process zone x nonlinearly increases with the external load specified by the dimensionless parameter σ (Fig. 3a). The dependence of the phase angle ψ of stresses in the zone on σ is less pronounced but also slowly increases with the absolute value of the load (Fig. 3b). The crack-tip opening displacement in the process zone $\delta u'$ weakly depends on the dimensionless parameter of loading σ (Fig. 3c).

In Fig. 4, we present the dependences of the parameters of the process zone on the phase angle of external loading φ. According to Fig. 4a, for a more plastic binding material $(n > 1)$, the length of the zone first somewhat decreases as the phase angle of external loading increases, attains the minimum value, then increases, attains the maximum value, and again decreases. For a brittler binding material $(n < 1)$, the length of the zone increases with the phase angle of external loading, attains its maximum value, then decreases, attains the minimum value, and again increases. The phase angle of stresses in the process zone first increases simultaneously with the phase angle of external loading (Fig. 4b) but remains lower than the latter, then takes a negative value as a result of a jump, and finally, increases. The jump is caused by the change in the sign of tangential stresses in the process zone. The behavior of the crack-tip opening displacement as a function of the phase angle of loading turns out to be opposite as compared with the behavior of the length (Fig. 4c).

The parameters of the process zone are strongly affected by the ratio of the tensile and shear strengths of the binding material $n = \sigma_0/\tau_0$. As follows from Fig. 5a, the length of the process zone increases with tensile strength for the constant parameter of loading. The phase angle of stresses in the zone for $\varphi \le 0$ is negative and rapidly decreases as n increases. At the same time, for $\varphi > 0$, it is positive and increases with n (Fig. 5b). According to Fig. 5c, the crack-tip opening displacement decreases as the tensile strength increases for the constant parameter of loading.

As a result of the Mellin inverse transformation of the transforms of stresses determined in solving the problem and applying the residue theorem, we can find the expansion of stresses in asymptotic series in the vicinity of the crack tip after the formation of the process zone. The principal terms of the expansion are determined by poles of the matrix coefficient $\mathbf{G}(p)$ of Eq. (8) within the interval $-1 < \operatorname{Re} p < 0$. These poles correspond to the zeros of the function $D(-1-p)$ in the same interval. According to the statement of the problem, these zeros determine the exponents of singularity of the stresses near the crack tip in the absence of the process zone. As already indicated, the equation $D(-1-p) = 0$ has complex roots for the kink angles close to $\alpha_0 = 180°$. Thus, unlike the available models of the process zone with discontinuity of a single component of displacements and the Tresca–Saint-Venant prefracture condition [3–6], in the analyzed cohesive model with the Mises–Hill condition, the formation of the process zone does not change the initial oscillatory character of singularity at the crack tip. This result is a consequence of the assumption accepted in the cohesive model about the discontinuity of both normal and tangential displacements. This makes the process zone similar to an interface crack whose faces suffer the action of constant tensile and shear forces. To remove the space oscillations of displacements and stresses in the vicinity of the corner point, it is necessary to take into account some additional factors affecting the stress-strain state near the crack tip. Thus, in particular, it is necessary to take into account the effect of contact of the faces and the formation of the domain of destruction of the material in the immediate vicinity of the corner point. The indicated domain should be characterized by the elevated level of strains in agreement with the complex model of interface cracks [7, 11].

CONCLUSIONS

By using the Wiener–Hopf method, we construct an approximate analytic solution of the problem of finding the parameters of a small-scale process zone in the binding material at the end of an interface crack originating from the corner point of the broken interface of two different homogeneous isotropic materials. The solution is constructed under the conditions of plane deformation within the framework of the Leonov–Panasyuk–Dugdale model and by using elements of the cohesive model. We deduce the equation for the length of the zone and the parameter of mixing of the modes of stresses in the zone, as well as the formulas for the crack-tip opening displacements. The possibility of application of these formulas to the determination of the ultimate loads in mixed loading modes is demonstrated. We also perform the numerical analyzes of parameters of the process zone and the qualitative analysis of their dependences on the kink angle in the interface of materials, external loads, and the characteristics of the binding material.

REFERENCES

1. F. D. Gakhov, *Boundary Value Problems*, Dover, New York (1990).
2. M. V. Dudyk, "Analytic solution of the plane problem of cohesive process zone in a binding material near the tip of an interface crack," *Visn. Odes. Nats. Univ. Mat. Mekh.*, **18**, Issue 4(20), 84–95 (2013); http://lib.onu.edu.ua/en/issledovaniya-v-matematike-i-mehanike/.
3. M. V. Dudyk and Yu. V. Dikhtyarenko, "Development of a prefracture zone from an interface crack at a corner point of an interface of two elastic media," *Mat. Met. Fiz.-Mekh. Polya*, **54**, No. 2, 103–114 (2011); ***English translation:*** *J. Math. Sci.*, **184**, No. 2, 121–135 (2012); https://doi.org/10.1007/s10958-012-0857-5.
4. M. V. Dudyk, Yu. V. Dikhtyarenko, and V. M. Dyakon, "Influence of the plasticity of a joining material on the kink of an interface crack at the corner point of the interface of media," *Fiz.-Khim. Mekh. Mater.*, **50**, No. 1, 45–52 (2014); ***English translation:*** *Mater. Sci.*, **50**, No. 1, 46–54 (2014).
5. A. A. Kaminsky, M. V. Dudik, and L. A. Kipnis, "On the direction of development of a thin fracture process zone at the tip of an interface crack between dissimilar media," *Prikl. Mekh.*, **42**, No. 2, 14–23 (2006); ***English translation:*** *Int. Appl. Mech.*, **42**, No. 2, 136–144 (2006).

6. A. A. Kaminsky, M. V. Dudik, and L. A. Kipnis, "Initial kinking of an interface crack between two elastic media," *Prikl. Mekh.*, **43**, No. 10, 28–41 (2007); ***English translation:*** *Int. Appl. Mech.*, **43**, No. 10, 1090–1099 (2007).
7. A. A. Kaminsky, M. V. Dudik, and L. A. Kipnis, "Investigation of the process zone near the tip of an interface crack in the elastic body in shear within the framework of the complex model," *Mat. Met. Fiz.-Mekh. Polya*, **57**, No. 4, 95–108 (2014); ***English translation:*** *J. Math. Sci.*, **220**, No. 2, 117–132 (2017); https://doi.org/10.1007/s10958-016-3171-9.
8. B. Noble, *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations*, Pergamon Press, London (1958).
9. V. V. Panasyuk, *Limit Equilibrium of Brittle Bodies with Cracks* [in Russian], Naukova Dumka, Kiev (1968).
10. V. V. Panasyuk and M. P. Savruk, "Model for plasticity bands in elastoplastic failure mechanics," *Fiz.-Khim. Mekh. Mater.*, **28**, No. 1, 49–68 (1992); ***English translation:*** *Sov. Mater. Sci.*, **28**, No. 1, 41–57 (1992).
11. M. V. Dudyk and L. A. Kipnis, "Model of the structure of the near tip area of interface crack in a piece-homogeneous elastic-plastic body," *Strength, Fract. Complex.*, **11**, No. 1, 31–50 (2018); DOI: 10.3233/SFC-180211.
12. Z.-H. Jin and C. T. Sun, "Cohesive zone modeling of interface fracture in elastic bi-materials," *Eng. Fract. Mech.*, **72**, No. 12, 1805–1817 (2005).
13. A. A. Khrapkov, *Wiener–Hopf Method in Mixed Elasticity Theory Problems*, B. E. Vedeneev VNIIG Publ. House, St.-Petersburg (2001).
14. K. Kishimoto, M. Omiya, and W. Yang, "Fracture mechanics of binding interface: a cohesive zone model," *Sens. Actuat.* A, **99**, 198–206 (2002).
15. M. J. Lee, T. M. Cho, W. S. Kim, B. C. Lee, and J. J. Lee, "Determination of cohesive parameters for a mixed-mode cohesive zone model," *Int. J. Adhes. Adhes.*, **30**, No. 5, 322–328 (2010).
16. J. R. Rice, "Elastic fracture mechanics concepts for interface cracks," *Trans. ASME. J. Appl. Mech.*, **55**, No. 1, 98–103 (1988).
17. C. T. Sun and Z.-H. Jin, "Modeling of composite fracture using cohesive zone and bringing models," *Compos. Sci. Technol.*, **66**, No. 10, 1297–1302 (2006).
18. V. Tvergaard, "Cohesive zone representations of failure between elastic or rigid solids and ductile solids," *Eng. Fract. Mech.*, **70**, No. 14, 1859–1868 (2003).
19. V. Tvergaard, "Influence of plasticity on interface toughness in a layered solid with residual stresses," *Int. J. Solids Struct.*, **40**, No. 21, 5769–5779 (2003).
20. V. Tvergaard and J. W. Hutchinson, "On the toughness of ductile adhesive joints," *J. Mech. Phys. Solids*, **44**, No. 5, 789–800 (1996).
21. B. H. Veitch and I. D. Abrahams, "On the commutative factorization of $n \times n$ matrix Wiener–Hopf kernels with distinct eigenvalues," *Proc. Math. Phys. Eng. Sci. (Proc. R. Soc. **A**)*, **463**, No. 2078, 613–639 (2007).